METHODS OF THEORETICAL PHYSICS

Fock–Schwinger Method in the Case of Different Masses

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The result of removing heavy particles with different mass from the theory spectrum can be described at low energy by the effective action, which is an asymptotic series in inverse powers of the mass. It is demonstrated how the leading terms of the asymptotic series can be calculated on the basis of the Fock–Schwinger proper time method. Theory with broken $U(3) \times U(3)$ chiral symmetry is considered to illustrate this approach. It is shown that different masses lead to the formation of a set of vertices describing the effects of the explicit breaking of flavor symmetry, which are usually not revealed in a less rigorous consideration of the problem. The result can be used in the study of theories with spontaneous symmetry breaking and in the development of effective theories of the Standard Model.

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The concept of proper time was introduced in the quantum theory by Fock [1, 2] and Schwinger [3]. Combined with the background field method, this allowed DeWitt to develop an explicitly covariant approach to gauge field theory [4] and quantum gravity [5]. The method is applicable in any field theory, but it is of special significance when the explicit covariance of calculations at all stages should be ensured. In all these cases and in many others [6, 7], it is necessary to calculate the determinant of a positively defined elliptic operator describing quadratic fluctuations of quantum fields in the external background field. The result is an asymptotic expansion of the effective action in powers of the proper time with the Seeley–DeWitt coefficients $a_n(x, y)$, which depend on background fields. It is noteworthy that each term of the expansion is invariant under the internal symmetry group because of the general covariance of the formalism. Now, the asymptotic coefficients a_n are well known up to $n = 5$ for an arbitrary Laplace-type operator, and the details of their calculations can be found, e.g., in [6–8].

When quantum fields have large and equal masses m , it is easy to transfer from an expansion in powers of the proper time to an expansion in inverse powers of the mass, which is valid when all background fields and their derivatives are small compared to the mass of the quantum field. In this case, the asymptotic coefficients a_n do not change. Such long-wavelength ($\lambda \gg 1/m$) expansion allows one to obtain an effective action describing the leading low-energy effect from virtual contributions of heavy states. This scenario often occurs in theories with spontaneous symmetry

breaking or in theories with heavy virtual particles. A typical example of theories of the first type is the Nambu–Jona-Lasinio model [9, 10], where the ground state in the strong-coupling regime is separated by a gap from the nucleons, which are excited quasiparticles. In the quark interpretation of the model, it is possible to obtain a low-energy effective meson action generated by chirally symmetric four-quark interactions [11]. The proper time expansion method is very useful here [12, 13]. Examples of the second type arise, in particular, in the effective extension of some field theory X with the symmetry group G to another theory X' including heavy degrees of freedom belonging to some representation of the group G . Currently, such theories are being actively studied in the context of extending the standard model of electroweak interactions [14].

In the studies cited above, it is often necessary to develop an effective theory arising from the integration of one-loop contributions of heavy states with different masses $M = \text{diag}(m_1, m_2, \dots, m_l)$. In this specific case, the complete factorization of M^2 in the heat kernel is impossible, since M does not commute with the rest of the elliptic operator. As a result, the asymptotic series has a more complex structure [15–17]. The mathematical complexity of this problem is often reduced by simplifications that usually do not have serious argumentation [13].

In [18–20], an alternative method for obtaining the effective action in the theory with heavy states of different masses was proposed, based on the rearrangement of the asymptotic series in powers of the proper

time. Resummation takes into account the difference in the masses of heavy particles. The method is especially efficient in the case of an almost degenerate matrix M . The advantage of this approach is that the coefficients of the modified expansion are related to the standard Seeley–DeWitt coefficients [21].

In this work, the above method is significantly developed. Specifically, the procedure for resumming the asymptotic series, which is based on the recurrent (i.e., step-by-step) selection of the final contribution depending on the difference between the squares of the masses, is not used. This procedure makes it possible to factorize the symmetric (with respect to mass permutation) part of the contribution. In this case, the antisymmetric part is expanded in a series in powers of the difference between the squares of the masses and modifies the Seeley–DeWitt coefficients. The rejection of this elegant idea introducing additional approximations makes the connection with the standard Seeley–DeWitt coefficients more complicated. As will be shown, the new expansion coefficients b_n depend not only on the background fields but also on the masses, and the latter dependence has a non-polynomial (logarithmic) form. In the limit of equal masses, b_n becomes a_n . An interesting feature of the proposed approach is the possibility of relating the mass component of the coefficients b_n to the Feynman integrals that generate them. Thus, each of the coefficients b_n contains complete information on the finite contributions of the related one-loop diagrams.

For definiteness, a theory with $U(3) \times U(3)$ chiral symmetry explicitly and spontaneously broken down to the subgroup $U(1)$ is considered. It is known that the group $U(3)_L \times U(3)_R$ corresponds to the symmetry of the QCD Lagrangian in the ideal world of massless quarks. In the real world with nonzero bare masses of current quarks, chiral symmetry is explicitly broken. At low energies, because of dynamic chiral symmetry breaking, a gap appears in the spectrum of fermions, and quarks become heavier and acquire different masses, which makes it possible to construct an effective low-energy meson Lagrangian in the form of $1/M^2$ series. The method described below can be used as the basis for a mathematically rigorous approach to solving this problem within the extended Nambu–Jona-Lasinio model.

The material is presented as follows. First, the problem is formulated. Then, the main idea of calculations is discussed and the main results are presented. Numerous details of the calculations will be presented in a separate detailed publication.

The logarithm of the formal determinant of a second-order self-adjoint elliptic operator describes the one-loop radiative corrections to the classical theory. Further, only the real part of the effective action, which is calculated from the determinant of the Dirac operator D in background scalar s , pseudoscalar p ,

vector v_μ , and axial-vector a_μ fields, is of interest. The proper time method cannot be applied directly to fermions, since the Dirac operator is a first-order elliptic operator, and its spectrum is unbounded from both above and below. Instead, it is appropriate to consider the functional

$$W_E = -\ln |\det D_E| \\ = \frac{1}{2} \int_0^\infty \frac{dt}{t} \rho(t, \Lambda^2) \text{Tr} \left(e^{-t D_E^\dagger D_E} \right), \quad (1)$$

representing the real part of the one-loop effective action in Euclidean space¹ in the form of an integral over the proper time t , where Tr denotes the trace over color, flavor, and gamma matrices, as well as integration over coordinates. The integral diverges at the lower limit; for this reason, a regulator $\rho(t, \Lambda^2)$ is introduced, where Λ is a cutoff. The Dirac operator has the form

$$D_E = i\gamma_\alpha d_\alpha - M + \Sigma, \quad (2)$$

where $d_\alpha = \partial_\alpha + i\Gamma_\alpha$, $\Gamma_\alpha = v_\alpha + \gamma_{5E} a_\alpha$, $\Sigma = s + i\gamma_{5E} p$, $\alpha = 1, 2, 3, 4$, and $M = \text{diag}(m_1, m_2, m_3)$ is the heavy quark mass matrix. Background fields take values in the Lie algebra of the $U(3)$ group, e.g., $s = s_a \lambda_a$, etc., $\lambda_a = (\lambda_0, \lambda_i)$, where $\lambda_0 = \sqrt{2/3}$ and λ_i are eight Hermitian $SU(3)$ Gell-Mann matrices. The dependence on external fields in D_E after transition to the Hermitian operator

$$D_E^\dagger D_E = M^2 - d_\alpha^2 + Y \quad (3)$$

is included in Y and the covariant derivative d_α .

To advance in the evaluation of Eq. (1), the Schwinger technique of a fictitious Hilbert space described in [6] is used. It allows one to represent the effective action as an integral over 4-momenta k_α

$$W_E = \frac{1}{2} \int d^4 x \int \frac{d^4 k}{(2\pi)^4} e^{-k^2} \\ \times \int_0^\infty \frac{dt}{t^3} \rho(t, \Lambda^2) \text{tr} \left(e^{-t(M^2 + A)} \right), \quad (4)$$

where $A = -d_\alpha^2 - 2ik\partial/\sqrt{t} + Y$. Since A contains derivatives with respect to coordinates, it is necessary to clarify the meaning of the operation tr . Since the space $x \in \mathbb{R}^4$ has no boundaries, integration by parts is possible. However, this operation will be unambiguous only if the trace does not change during cyclic permutation of its x -dependent elements. Taking into account the presence of derivatives in A , the condition

¹ All calculations are carried out in Euclidean space, as indicated by subscript E. The analytic continuation of the result obtained into the Minkowski space is well known and is not discussed here.

in (4) is fulfilled if complete symmetrization of the trace is assumed; i.e., tr in such cases is understood as

$$\text{str}(A_1 A_2 \dots A_n) = \frac{1}{n_{\text{cycl. perm.}}} \sum \text{tr}(A_1 A_2 \dots A_n). \quad (5)$$

To distinguish the leading contributions in the expansion in $1/M^2$, it is enough to take into account terms of at most t^4 order, which under the flavor trace (indicated by f) have the form

$$\begin{aligned} \text{tr}_f \left(e^{-(M^2 + A)} \right) &= \sum_{i=1}^3 c_i(t) - t \sum_{i=1}^3 c_i(t) \text{tr}_f A_i \\ &+ \frac{t^2}{2!} \sum_{i,j} c_{ij}(t) \text{tr}_f (A_i A_j) - \frac{t^3}{3!} \sum_{i,j,k} c_{ijk}(t) \text{tr}_f (A_i A_j A_k) \\ &+ \frac{t^4}{4!} \sum_{i,j,k,l} c_{ijkl}(t) \text{tr}_f (A_i A_j A_k A_l) + \mathcal{O}(t^5). \end{aligned} \quad (6)$$

Here, $A_i = E_i A$, where 3×3 matrices $(E_i)_{nm} = \delta_{in} \delta_{im}$, and coefficients $c_{i_1 i_2 \dots i_n}(t)$ are completely symmetric with respect to index permutation and are well known [19]. They depend on masses and proper time. If the masses are degenerate, they all coincide, i.e., $c_i(t) = c_{ii}(t) = c_{iii}(t) = c_{iiii}(t) = e^{-tm_i^2}$.

The substitution of Eq. (6) into Eq. (4) and integration over momenta and proper time give the expression

$$W_E = \frac{N_c}{32\pi^2} \int d^4 x \sum_{n=0}^{\infty} \text{tr}_{L_f} b_n(x, x), \quad (7)$$

where $N_c = 3$, and tr_{L_f} denotes the trace over the γ and λ_i matrices. The coefficients $b_n(x, x)$ depend on the external fields and quark masses; i.e., they contain information about both meson vertices and corresponding coupling constants. In the case of equal masses, the dependence on m is factorized in the form

of the integrals $J_{n-1}(m^2)$ [19], and the meson vertices are collected in the Seeley–DeWitt coefficients, more accurately in their limit value for coinciding arguments $a_n(x, x)$. For large m values, the coefficients $b_n(x, x)$ demonstrate the same asymptotic behavior as $J_{n-1}(m^2)$, i.e., $b_n \sim m^{-2(n-2)}$.

Here, only the leading coefficients, namely, b_1 and b_2 , will be considered. The case $n = 0$ is of no interest since b_0 contains no background fields and is removed from the effective action. The coefficients with $n \geq 3$ tend to zero in the limit of infinite masses, which explains the approximation made.

For the convenient representation of the result, along with the usual matrix multiplication, a nonstandard often called Hadamard product [22] is also used,

which is determined by the multiplication of the corresponding elements of the matrices

$$(A \circ B)_{ij} = A_{ij} B_{ij} \quad (8)$$

without summation over repeated indices. This product, unlike the standard one, is commutative, but the associative and distributive properties are retained.

Now, the result for the coefficient b_1 can be written as

$$b_1 = -J_0 \circ Y, \quad (9)$$

where $(J_0)_{ij} = \delta_{ij} J_0(m_i^2)$ is a diagonal matrix with

$$J_0(m_i^2) = \int_0^\infty \frac{dt}{t^2} \rho(t, \Lambda^2) c_i(t). \quad (10)$$

This formula describes the contribution of a Feynman one-loop diagram, known as the “tadpole.” The regularization is chosen according to the problem under consideration; for the Nambu–Jona-Lasinio model, it is $\rho(t, \Lambda^2) = 1 - (1 + t\Lambda^2)e^{-t\Lambda^2}$.

Consider the second coefficient b_2 . The result is represented as

$$b_2 = \frac{1}{2} Y (J \circ Y) - \frac{1}{12} \Gamma^{\alpha\beta} (J \circ \Gamma^{\alpha\beta}) + \Delta b_2, \quad (11)$$

where $\Gamma_{\alpha\beta} = F_{\alpha\beta} + i[\Gamma_\alpha, \Gamma_\beta]$ is an antisymmetric tensor and $F_{\alpha\beta} = \partial_\alpha \Gamma_\beta - \partial_\beta \Gamma_\alpha$. $J_{ij} = J(m_i, m_j)$ is a symmetric 3×3 matrix, whose elements are logarithmically divergent parts (at $\Lambda \rightarrow \infty$) of Feynman self-energy diagrams with masses of virtual particles m_i and m_j . Expressions for similar contributions of triangular J_{ijk} and quadrangular J_{ijkl} diagrams also required below can be represented together with J_{ij} as

$$(J_{ij}, J_{ijk}, J_{ijkl}) = \int_0^\infty \frac{dt}{t} \rho(t, \Lambda^2) (c_{ij}(t), c_{ijk}(t), c_{ijkl}(t)). \quad (12)$$

In the theory under consideration, there are only three flavors of quarks; thus, at least two indices in J_{ijk} coincide. Note that formula (11) is the main result of this work.

In the limit of equal masses, the first two terms in Eq. (11) are transformed to the well-known result [6], and the third term vanishes; i.e., it represents only contributions related to the explicit chiral symmetry breaking. Without this term, the description of the exact flavor symmetry breaking is incomplete. It can be written in the form of four terms

$$\Delta b_2 = \Omega_Y + \sum_{n=0}^2 \Omega^{(n)}, \quad (13)$$

where the sum of the terms linear in Y is marked as Ω_Y , and the contribution of vector particles is divided into

three groups $n = 0, 1, 2$ according to the number of entering derivatives.

The part linear in Y consists of two terms

$$\Omega_Y = AY + \frac{i}{3}(R \circ \Gamma^\alpha) \tilde{\partial}^\alpha Y, \quad (14)$$

where R and A are 3×3 matrices with the elements

$$R_{ij} = -\frac{1}{2}(J_{ij} - J_{jji}), \quad (15)$$

$$A_{ii} = \sum_{j \neq i} (J_{ii} - J_{ij} + R_{ij}) \Gamma_{ij}^\alpha \Gamma_{ji}^\alpha,$$

$$A_{ij} = (J_{ij} - J_{ijk}) \Gamma_{ik}^\alpha \Gamma_{kj}^\alpha \quad (16)$$

$$- R_{ij} (\Gamma_{ij}^\alpha \Gamma_{jj}^\alpha - \Gamma_{ii}^\alpha \Gamma_{ij}^\alpha)_{i \neq j \neq k},$$

and the two-sided derivative is defined as the difference $Y \tilde{\partial}^\alpha \Gamma^\alpha \equiv Y \partial^\alpha \Gamma^\alpha - (\partial^\alpha Y) \Gamma^\alpha$. To avoid misunderstandings, we emphasize that repeated flavor indices do not imply summation over them. Here and below, summation is carried out only if the summation symbol is explicitly written.

Now consider the terms with two derivatives

$$\Omega^{(2)} = \frac{1}{24} [F^{\alpha\beta} (T \circ F^{\alpha\beta}) + 6(\partial\Gamma)(T \circ \partial\Gamma)]. \quad (17)$$

The first term makes an additional contribution to the already known kinetic terms of the effective Lagrangian. The symmetric matrix T has the form $T_{ij} = J_{ij} - J_{ijj}$. The expression $\partial\Gamma$ implies summation over the omitted indices $\partial^\alpha \Gamma^\alpha$. Since the vector fields are transverse, the second term in Eq. (17) vanishes on the mass shell of these states. Therefore, it can be omitted.

The terms with one derivative can be summarized in the expression

$$\Omega^{(1)} = \frac{i}{3} C^{\alpha\beta} F^{\alpha\beta} + \frac{i}{6} \delta_{\alpha\beta\gamma\sigma} (K \circ \partial^\alpha \Gamma^\beta) E^{\gamma\sigma} + \frac{3i}{4} (T \circ \partial^\alpha \Gamma^\beta) L^{\beta\alpha}, \quad (18)$$

which represents the effective three-particle vertices describing the local interaction of vector and axial-vector fields. The symbol $\delta_{\alpha\beta\gamma\sigma} = \delta_{\alpha\beta} \delta_{\gamma\sigma} + \delta_{\alpha\gamma} \delta_{\beta\sigma} + \delta_{\alpha\sigma} \delta_{\beta\gamma}$ is a totally symmetric tensor composed of the product of two Kronecker deltas. The diagonal and off-diagonal elements of the matrix $C^{\alpha\beta}$ have the form

$$C_{ii}^{\alpha\beta} = \sum_{j \neq i} \left(J_{ii} - J_{ij} + \frac{1}{4} R_{ij} + \frac{3}{4} T_{ij} \right) \Gamma_{ij}^\alpha \Gamma_{ji}^\beta, \\ C_{ij}^{\alpha\beta} = (J_{ij} - J_{123}) \Gamma_{ik}^\alpha \Gamma_{kj}^\beta \Big|_{i \neq j \neq k} + \frac{1}{4} R_{ij} (L_+^{\alpha\beta})_{ij} \\ + \frac{3}{4} T_{ij} (L_-^{\alpha\beta})_{ij}, \quad (19)$$

respectively, where

$$(L_\pm^{\alpha\beta})_{ij} = (\Gamma_{ii}^\alpha \pm \Gamma_{jj}^\alpha) \Gamma_{ij}^\beta. \quad (20)$$

Note that the elements of the matrix Γ_{ij}^α commute with each other, since the identity and γ_5 matrices included in them are commutative. The antisymmetric matrix K is defined by the expression

$$K_{ij} = (J_{jik} - J_{ijk})_{i \neq j \neq k}. \quad (21)$$

The tensor $E^{\gamma\sigma}$ also has no diagonal elements, although it is not antisymmetric

$$E_{ii}^{\gamma\sigma} = 0, \quad E_{ij}^{\gamma\sigma} = \Gamma_{ik}^\gamma \Gamma_{kj}^\sigma, \quad (i \neq j \neq k). \quad (22)$$

Finally, the contribution of terms without derivatives, i.e., terms $\propto \Gamma^4$ should be considered. Since there are many such terms (78 independent combinations of Γ), it is convenient to represent the result in the form of explicit sums over flavor indices, rather than collecting them in the matrix form. Then,

$$\begin{aligned} \text{tr}_f \Omega^{(0)} = & \sum_{i < j} \left\{ (\Gamma_{ij} \Gamma_{ji})^2 \left[\frac{2}{3} (J_{ii} + J_{jj}) - \frac{4}{3} J_{ij} \right] \right. \\ & \left. + (\Gamma_{ij} \Gamma_{ij}) (\Gamma_{ji} \Gamma_{ji}) \frac{1}{6} (2J_{ij} - J_{ii} - J_{jj}) \right\} \\ & + \sum_{\substack{i \neq j \neq k, \\ j < k}} \left\{ (\Gamma_{ij} \Gamma_{ji}) (\Gamma_{ik} \Gamma_{ki}) \left[\frac{1}{3} (J_{jk} + 2J_{ijk}) + J_{ii} - J_{ij} - J_{ik} \right] \right. \\ & + (\Gamma_{ji} \Gamma_{ik}) (\Gamma_{ki} \Gamma_{ij}) \left[\frac{1}{3} (J_{ii} + 2J_{ijk}) + J_{jk} - 2J_{ijk} \right] \\ & + [(\Gamma_{ii} \Gamma_{jk}) (\Gamma_{ij} \Gamma_{ki}) + (\Gamma_{ik} \Gamma_{ji}) (\Gamma_{kj} \Gamma_{ii})] \\ & \times \frac{1}{3} (2J_{ijk} - J_{ij} - J_{ik}) \\ & + (\Gamma_{ij} \Gamma_{ik}) (\Gamma_{ki} \Gamma_{ji}) \frac{1}{3} (2J_{ijk} - J_{ii} - J_{jk}) \left. \right\} \\ & + \sum_{i \neq j \neq k} \left\{ [(\Gamma_{ii} \Gamma_{ij}) (\Gamma_{jk} \Gamma_{ki}) + (\Gamma_{ik} \Gamma_{kj}) (\Gamma_{ji} \Gamma_{ii})] \right. \\ & \times \frac{1}{3} (J_{ik} + 2J_{ijk} - J_{ijk}) \\ & + \left[(L_-^{\alpha\alpha})_{ij} (\Gamma_{jk} \Gamma_{ki}) + (\Gamma_{ij} \Gamma_{ji}) (\Gamma_{ik} \Gamma_{ki}) \right] R_{ij} \left. \right\} \\ & + \sum_{i \neq j} \left(\Gamma_{ii}^\alpha \Gamma_{ii}^\beta \Gamma_{ij}^\gamma \Gamma_{ji}^\sigma - \Gamma_{ii}^\alpha \Gamma_{ij}^\beta \Gamma_{jj}^\gamma \Gamma_{ji}^\sigma - \frac{1}{2} \Gamma_{ij}^\alpha \Gamma_{ji}^\beta \Gamma_{ij}^\gamma \Gamma_{ji}^\sigma \right) \\ & \times \frac{1}{3} \delta_{\alpha\beta\gamma\sigma} T_{ij}. \end{aligned} \quad (23)$$

Here, the expression in the parentheses is considered as $(\Gamma_{ij} \Gamma_{ji}) \equiv \Gamma_{ij}^\alpha \Gamma_{ji}^\alpha$. In the limit of equal masses of fermions, Eq. (23) is zero, since all J -dependent coefficients vanish.

In conclusion, the expressions obtained for the coefficients $b_1(x, x)$ and $b_2(x, x)$ can be used as the

basis for constructing a series of effective theories that arise as a result of the exclusion of heavy states of different mass. The most direct application is to use them to analyze explicit $SU(3)$ and $SU(2)$ symmetry breaking in Nambu–Jona-Lasinio models. As already noted, numerous studies of these issues available in the literature are incomplete, since the coefficient b_2 contains many previously unconsidered vertices. Although a particular example is considered in this work, the proposed approach can be easily extended to the case of an arbitrary internal symmetry group. This is a direct consequence of the generality of the Fock–Schwinger method underlying the performed calculations.

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