The muon pair production in the process $e^+e^- \rightarrow e^+e^-\mu^+\mu^-$ through the two-photon mechanism at high energies: transverse momentum, rapidity and angular distributions

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Abstract In the present paper, we investigate the muon pair production in the interaction between two quasi-real photons in e^+e^- collision. We calculate the total and differential cross section of the process $\gamma\gamma \rightarrow \mu^+\mu^-$ at a beam energy of photons from 3 to 40 GeV in the center-of-mass and for different values of the muon transverse momentum, the muon rapidity and the muon angle. In the $\gamma\gamma \rightarrow \mu^+\mu^-$ process we also take the contribution of the Box+Vertex correction+Soft photon emission+Vacuum polarization diagrams. We also study the cross section of the $e^+ + e^- \rightarrow e^+ + e^- + \mu^+ + \mu^-$ process as a function of the e^+e^- center-of-mass energy $\sqrt{s}$ in the region $5 \text{ GeV} \leq \sqrt{s} \leq 209 \text{ GeV}$ using the two-photon mechanism. The obtained results are in satisfactory agreement with experimental data.

1 Introduction

The microworld is supposed to be described by a consistent mathematical theory – the standard model. The main problems in particle physics are the confirmation of this model, which has been up to now successfully tested at colliders and accelerators, and the search for possible physics beyond this model [1–4].

In particle physics we are continually searching for the most fundamental method of description of the interactions that govern the material world.

The main task in physics is to describe all physical phenomena with a few basic principles. Elementary particle

physics is used to explain the many faces of the Universe by means of quarks and leptons, and their interactions.

Quantum electrodynamics (QED) and the standard model (SM) have already been successful in remodeling the electromagnetic and weak interactions between charged particles, intermediate bosons and photons.

Electron–positron, photon–photon and photon–electron scattering processes in QED and the SM at colliders of high energies attract a lot of attention of both theorists and experimentalists. The existence of the photon linear collider gives the foundation for creation an e^+e^- linear collider [5–12, 14, 15, 15–18], and as a result, the study of processes at other linear and circular colliders, and also at hadron colliders, will be useful. These accelerators with high-energy beams colliding e^+e^- , $\gamma\gamma$, γe , $\mu^+\mu^-$ were widely used to study the fundamental interactions. It is possible that some QED processes played an important role at these colliders, and the cross section did not decrease with increasing energy.

The laws of physics in the nuclear domain are derived mainly from the analysis of the outcomes of particle collisions at high-energy.

The study of the possible photon–photon scattering at a high-energy electron–positron linear collider is one of the main physics goals.

There are experiments that measured the production of a muon pair, which is fairly well described in QED. This fact makes important to study the process of muon pair production in lepton–lepton collisions.

The production of a muon pair with large transverse momentum in photon–photon processes plays a fundamental role in electron–positron collisions. Lepton pair production in photon–photon scattering is often a background process for possible manifestation of the new physics.

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Fig. 1 The Feynman diagrams for the muon pair production in $\gamma\gamma$ collisions in the Born approximation

It should be noted that two-photon processes $\gamma\gamma \rightarrow X$ were investigated in details by many authors and are extremely important laboratory for modern high-energy physics [19–21]. Experimentally, they are studied on a collision e^+e^- beams interacting with virtual photons, which they are emitted by the initial particles: e.g. in the reaction $e^+e^- \rightarrow e^+e^-\gamma^*\gamma^* \rightarrow e^+e^-X$.

By the other hand we have the reaction $e^+e^- \rightarrow e^+e^-\ell^+\ell^-$, where ℓ can be any charged lepton, which arises up to $O(\alpha^4)$. It is investigated in detail [19].

Consequently the process $e^+e^- \rightarrow e^+e^-\mu^+\mu^-$ provides a good way to test QED at the level of perturbation theory of order of α^4 .

LEP collider provides very high precision of particle detection and thus can be used for precise study of $e^+e^- \rightarrow e^+e^-\mu^+\mu^-$, processes [22,23].

Electron–positron colliders are ideal to study the characteristics of lepton pair productions, because the production mechanisms include both: electromagnetic and the electroweak interactions. Therefore, the cross-sections can be estimated quite reliably, as also the final-state distributions.

Up to now many studies have been carried out for processes of photon–photon collisions, both theoretical and experimental, which have made important contributions to these areas [19,20,24,25,27–48,50–93].

To study the process $e^+ + e^- \rightarrow e^+ + e^- + \mu^+ + \mu^-$, several experiments were carried out [92–99], and theoretical calculations within QED were made. There are many groups which made relevant calculations [19,100,101,103,104,106,108–111,113–128].

A large number of studies of the photon–photon collision processes have been performed on the detectors at the LEP, BELLE, BABAR, VEPP, and other e^+e^- colliders. The process $e^+ + e^- \rightarrow e^+ + e^- + \mu^+ + \mu^-$ at $161 \text{ GeV} < \sqrt{s} < 209 \text{ GeV}$ has been studied on the L3 detector. In this process, the muon pair invariant mass for the photon–photon collisions was measured in the range $3 \text{ GeV} < W_{\gamma\gamma} < 40 \text{ GeV}$ [92].

Our aim in this work is to study the characteristics of the photon–photon interaction process, and take into account the contributions of the Box + Vertex corrections + Soft photon emission + Vacuum polarization diagrams with the radiative

corrections (RC) to this process. Also through this process to study $e^+e^- \rightarrow e^+e^-\mu^+\mu^-$ from different kinematic variables of experimental data from the LEP. We think that the results obtained by us will be useful in the context of the linear collider, and also at other colliders.

In this paper, the first issue is to calculate the total and differential cross sections with taking into account the vertex corrections, and soft photon emission, and vacuum polarization for the processes $\gamma\gamma \rightarrow \mu^+\mu^-$ and $e^+ + e^- \rightarrow e^+e^-\mu^+\mu^-$ with $\sqrt{s}$, p_T , rapidity (y) and angle within the framework of QED and the SM.

The second point is to study the $e^+ + e^- \rightarrow e^+ + e^- + \mu^+ + \mu^-$ process initiated by electron–positron collisions, and in this way to illustrate some fundamental concepts and issues involving precisely QED and SM.

We want to see that the center-of-mass energies are in the range of the LEP collider $161 \text{ GeV} \leq \sqrt{s} \leq 209 \text{ GeV}$, and the scattering angle is in the range $15^\circ < \theta < 165^\circ$.

2 The process $\gamma\gamma \rightarrow \mu^+\mu^-$

In the present paper, in electron–positron collisions, the massive muon-pair production proceeds through photon–photon collisions, i.e. through the two-photon mechanism.

To study the $e^+ + e^- \rightarrow e^+ + e^- + \mu^+ + \mu^-$ process, at the first stage we must consider the production of muon pairs in $\gamma\gamma \rightarrow \mu^+\mu^-$ collisions. Therefore, in this section we want to discuss the $\mu^+\mu^-$ production in photon–photon collisions.

This process can be represented as the following physical picture: at leading order, two photons interact and produce a measured final state of mass m_μ and transverse momentum p_T .

The process is written in the form:

$$\gamma(p_1) + \gamma(p_2) \rightarrow \mu^-(k_1) + \mu^+(k_2). \quad (1)$$

The Feynman diagrams of the process (1) in the Born approximation are shown in Fig. 1.

The Box diagrams of the process (1) are shown in Fig. 2.

Fig. 2 Box-type of the Feynman diagrams contributing to the muon pair production in the process $\gamma\gamma \rightarrow \mu^+\mu^-$ at the one-loop level

The kinematics of the process (1) can be described in terms of the following Mandelstam variables [129]:

$$\begin{aligned} s &= (p_1 + p_2)^2 = (k_1 + k_2)^2; \\ t &= (p_1 - k_1)^2 = (p_2 - k_2)^2; \\ u &= (p_1 - k_2)^2 = (p_2 - k_1)^2, \end{aligned} \quad (2)$$

where p_1 and p_2 are the 4-momenta of the photons in the initial state; k_1 and k_2 are the 4-momenta of the μ^- and μ^+ in the final state. In this process, the created muons have masses m_μ ; then it follows from (2) that the sum of Mandelstam variables, which we will use, can be written in the form

$$s + t + u = 2m_\mu^2. \quad (3)$$

The matrix elements corresponding to the Feynman diagrams in the Born approximation (Fig. 1) can be written in the form

$$\begin{aligned} i\mathcal{M}_1 &= -\bar{u}(k_1, m_\mu)(-ie\gamma_\alpha) \frac{i(\hat{p}_2 - \hat{k}_2 + m_\mu)}{(k_2 - p_2)^2 - m_\mu^2} \\ &\quad \times (-ie\gamma_\beta)v(k_2, m_\mu)\varepsilon_\alpha(p_1)\varepsilon_\beta(p_2); \\ i\mathcal{M}_2 &= -\bar{u}(k_1, m_\mu)(-ie\gamma_\beta) \frac{i(\hat{k}_1 - \hat{p}_2 + m_\mu)}{(p_2 - k_1)^2 - m_\mu^2} \\ &\quad \times (-ie\gamma_\alpha)v(k_2, m_\mu)\varepsilon_\alpha(p_1)\varepsilon_\beta(p_2); \end{aligned} \quad (4)$$

The matrix elements corresponding to the box diagrams (Fig. 2) can be written as follows:

$$\begin{aligned}
i\mathcal{M}_3 &= -\frac{i(2ie)^4}{16\pi^4} \int d^4q_1 \frac{\bar{u}(k_1)\gamma_\mu(\hat{k}_1 + \hat{k}_2 - \hat{p}_2 - \hat{q}_1 + m_\mu)\gamma_\alpha(-\hat{q}_1 + m_\mu)}{(q_1^2 - m_\mu^2)((p_2 + q_1 - k_1 - k_2)^2 - m_\mu^2)} \\
&\quad \cdot \frac{\gamma_\beta(-\hat{p}_2 - \hat{q}_1 + m_\mu)\gamma_\nu v(k_2)}{((p_2 + q_1)^2 - m_\mu^2)(p_2 + q_1 - k_2)^2} \\
&\quad \cdot g^{\mu\nu} \varepsilon_\alpha(p_1) \varepsilon_\beta(p_2); \\
i\mathcal{M}_4 &= -\frac{i(2ie)^2}{16\pi^4} \int d^4q_1 \frac{\bar{u}(k_1)\gamma_\mu(\hat{k}_1 + \hat{k}_2 - \hat{p}_2 - \hat{q}_1 + m_\mu)\gamma_\alpha(-\hat{q}_1 + m_\mu)}{(q_1^2 - m_\mu^2)((p_2 + q_1 - k_1 - k_2)^2 - m_\mu^2)} \\
&\quad \cdot \frac{\gamma_\beta(-\hat{p}_2 - \hat{q}_1 + m_\mu)\gamma_\nu v(k_2) \cdot g^{\mu\nu} \cdot \varepsilon_\alpha(p_1) \varepsilon_\beta(p_2)}{((p_2 + q_1)^2 - m_\mu^2)((p_2 + q_1 - k_2)^2 - M_Z^2)} \cdot \left(\frac{ie(-\frac{1}{2} + \sin^2\theta_W)}{\cos\theta_W \sin\theta_W} + \frac{iesin\theta_W}{\cos\theta_W} \right)^2; \\
i\mathcal{M}_5 &= -\frac{ie^2}{16\pi^4} \int d^4q_1 \frac{\bar{u}(k_1)\gamma_\sigma(\hat{p}_2 + \hat{q}_1 - \hat{k}_2)\gamma_\kappa \cdot v(k_2) \cdot g^{\mu\nu} g^{\rho\sigma} g^{\lambda\kappa}}{(q_1^2 - M_W^2)((p_2 + q_1)^2 - M_W^2)(p_2 + q_1 + k_2)^2((p_2 + q_1 - k_1 - k_2)^2 - M_W^2)} \\
&\quad \cdot \left(\frac{ie}{\sqrt{2}\sin\theta_W} \right)^2 [(-p_1 - q_1)^\rho g^{\alpha\mu} + (p_1 - p_2 - q_1 + k_1 + k_2)^\mu g^{\alpha\rho} + (p_2 + 2q_1 - k_1 - k_2)^\alpha g^{\mu\rho}] \\
&\quad \cdot [(-p_2 + q_1)^\lambda g^{\beta\nu} + (2p_2 + q_1)^\nu g^{\beta\lambda} + (-p_2 - 2q_1)^\beta g^{\nu\lambda}] \\
&\quad \cdot \varepsilon_\alpha(p_1) \varepsilon_\beta(p_2); \tag{5}
\end{aligned}$$

$$\begin{aligned}
i\mathcal{M}_6 &= -\frac{i(2ie)^4}{16\pi^4} \int d^4q_1 \frac{\bar{u}(k_1)\gamma_\mu(\hat{p}_2 + \hat{q}_1 + m_\mu)\gamma_\beta(\hat{q}_1 + m_\mu)\gamma_\alpha}{(q_1^2 - m_\mu^2)((p_2 + q_1 - k_1 - k_2)^2 - m_\mu^2)} \\
&\quad \cdot \frac{(\hat{p}_2 + \hat{q}_1 - \hat{k}_1 - \hat{k}_2 + m_\mu)\gamma_\nu v(k_2)}{((p_2 + q_1)^2 - m_\mu^2)(p_2 + q_1 - k_2)^2} \cdot g^{\mu\nu} \varepsilon_\alpha(p_1) \varepsilon_\beta(p_2); \\
i\mathcal{M}_7 &= -\frac{i(2ie)^2}{16\pi^4} \int d^4q_1 \frac{\bar{u}(k_1)\gamma_\mu(\hat{p}_2 + \hat{q}_1 + m_\mu)\gamma_\beta(\hat{q}_1 + m_\mu)\gamma_\alpha(\hat{p}_2 + \hat{q}_1 - \hat{k}_1 - \hat{k}_2 + m_\mu)}{(q_1^2 - m_\mu^2)((p_2 + q_1 - k_1 - k_2)^2 - m_\mu^2)((p_2 + q_1)^2 - m_\mu^2)} \\
&\quad \cdot \frac{\gamma_\nu v(k_2) \cdot g^{\mu\nu} \cdot \varepsilon_\alpha(p_1) \varepsilon_\beta(p_2)}{((p_2 + q_1 - k_2)^2 - M_Z^2)} \\
&\quad \cdot \left(\frac{ie(-\frac{1}{2} + \sin^2\theta_W)}{\cos\theta_W \sin\theta_W} + \frac{iesin\theta_W}{\cos\theta_W} \right)^2; \\
i\mathcal{M}_8 &= -\frac{ie^2}{16\pi^4} \int d^4q_1 \frac{\bar{u}(k_1)\gamma_\lambda(-\hat{p}_2 - \hat{q}_1 + \hat{k}_1)\gamma_\sigma \cdot v(k_2) \cdot g^{\mu\nu} g^{\rho\sigma} g^{\lambda\kappa}}{(q_1^2 - M_W^2)((p_2 + q_1)^2 - M_W^2)(p_2 + q_1 - k_1)^2((p_2 + q_1 - k_1 - k_2)^2 - M_W^2)} \\
&\quad \cdot \left(\frac{ie}{\sqrt{2}\sin\theta_W} \right)^2 [(-p_1 - q_1)^\rho g^{\alpha\mu} + (p_1 - p_2 - q_1 + k_1 + k_2)^\mu g^{\alpha\rho} + (p_2 + 2q_1 - k_1 - k_2)^\alpha g^{\mu\rho}] \\
&\quad \cdot [(-p_2 + q_1)^\kappa g^{\beta\nu} + (2p_2 + q_1)^\nu g^{\beta\kappa} + (-p_2 - 2q_1)^\beta g^{\nu\kappa}] \\
&\quad \cdot \varepsilon_\alpha(p_1) \varepsilon_\beta(p_2);
\end{aligned}$$

We consider the subprocess $\gamma\gamma \rightarrow \mu^+\mu^-$ in the leading order and also in the second order of perturbation theory. Then, the general matrix elements of the Feynman diagrams in the Born approximation in Fig. 1 can be written in the following form:

$$M_{Born} = \mathcal{M}_1 + \mathcal{M}_2. \tag{6}$$

The general matrix elements for the Box diagrams in Fig. 2 can be written as follows

$$M_{Box} = \mathcal{M}_3 + \mathcal{M}_4 + \mathcal{M}_5 + \mathcal{M}_6 + \mathcal{M}_7 + \mathcal{M}_8. \tag{7}$$

We do not take into account the higher order α^2 in the Box diagram.

Taking into account the mass of muons, $k_1^2 = k_2^2 = m_\mu^2$ (for the process $\gamma\gamma \rightarrow \mu^+\mu^-$), we can obtain a boundary

Fig. 3 The total cross section of the process $\gamma\gamma \rightarrow \mu^+\mu^-$ as a function of the $\gamma\gamma$ centre-of-mass energy. (for all contribution – Born+Box+VC+SPE+VP diagrams) The experimental data are from [92]

equation for the physical regions in the following form

$$tu = m_\mu^4. \quad (8)$$

For given s and muon mass and after solving two Eqs. (3) and (8), we can define the upper and lower bounds for t as:

$$t^{\max, \min} = \frac{-s + 2m_\mu^2}{2} \pm \frac{s}{2} \sqrt{1 - \frac{4m_\mu^2}{s}}. \quad (9)$$

Expressions for the square of the matrix element of the Born diagram and the terms $M_{\text{Born}}^+ \cdot M_{\text{Box}}$ are given in 6 (Appendix A). In our calculation, we have used the completeness relation of the sum over the photon polarizations in the initial state, replacing by the expression:

$$\sum_{\lambda} \varepsilon_{\mu}(p_1, \lambda) \varepsilon_{\nu}^*(p_1, \lambda) = -g_{\mu\nu}, \quad (10)$$

where $\varepsilon(p_1, \lambda)$ and $\varepsilon(p_2, \lambda)$ are the incoming photon polarization vectors.

The “master formula” for evaluating the cross section and kinematic distributions for the $\gamma(p_1) + \gamma(p_2) \rightarrow \mu^-(k_1) + \mu^+(k_2)$ process is

$$d\sigma = \frac{1}{2s} \int \frac{d^3\mathbf{k}_1}{(2\pi)^3 2E_1} \frac{d^3\mathbf{k}_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta^4 \times (k_1 + k_2 - p_1 - p_2) \overline{|M|^2}, \quad (11)$$

and the differential cross section for the $\gamma(p_1) + \gamma(p_2) \rightarrow \mu^-(k_1) + \mu^+(k_2) + \gamma(k)$ process is

$$d\sigma = \frac{1}{2s} \int \frac{d^3\mathbf{k}_1}{(2\pi)^3 2E_1} \frac{d^3\mathbf{k}_2}{(2\pi)^3 2E_2} \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega} (2\pi)^4 \delta^4 \times (k_1 + k_2 + k - p_1 - p_2) \overline{|M|^2}, \quad (12)$$

where M is the invariant matrix element.

In the phase volume, after integrating over $\mathbf{k}_2$ in (11), using the δ -function and after some simple transformation, we can get the differential cross section for the $\gamma\gamma \rightarrow \mu^+\mu^-$ process (1) in the center-of-mass as:

$$\frac{d\sigma}{dt} = \frac{1}{16\pi s^2} \beta_\mu \overline{|M|^2}, \quad (13)$$

where $\beta_\mu = \sqrt{1 - \frac{4m_\mu^2}{s}}$ and $\sqrt{s}$ is the total centre-of-mass energy of the colliding photons.

The total cross section for the subprocess in the photon–photon collisions (1) can be obtained by

$$\sigma(\gamma\gamma \rightarrow \mu^+\mu^-) = \int_{t^{\min}}^{t^{\max}} dt \frac{d\sigma}{dt}, \quad (14)$$

where $t^{\max, \min}$ are defined in (9).

We present the results of our numerical calculations in Figs. 3, 4, 5, 6, 7 and 9.

In accordance with the L3 experiment [92], we calculated the total cross section as the sum of all the contribution (Born + Box + Vertex correction + Soft photon emission + Vacuum polarization diagrams) of the $\gamma\gamma \rightarrow \mu^+\mu^-$ process (1) at the center-of-mass energy with in the range $3 \text{ GeV} \leq \sqrt{s} \leq 40 \text{ GeV}$ and plotted the dependence of the total cross section on $\sqrt{s}$. We plot this dependence in Fig. 3. It is shown that the cross section is slowly decreasing, and a comparison with experimental data shows that our result with a few experimental points are in good agreement, but we do not have full agreement.

Fig. 4 The Feynman diagrams for vertex contributions to the process $\gamma\gamma \rightarrow \mu^+\mu^-$

Fig. 5 Vertex radiative corrections

Fig. 6 The Feynman diagrams for soft photon emission to the process $\gamma\gamma \rightarrow \mu^+\mu^-$

Fig. 7 The Feynman diagrams for the vacuum polarization contributions to the process $\gamma\gamma \rightarrow \mu^+\mu^-$

2.1 Vertex corrections

In Fig. 4 are shown the diagrams corresponding to vertex contributions (vertex diagrams). A circle in them stands for photon contributions within the QED framework, and the notation for the 4-momenta is identical as in the Born diagrams in Fig. 1. Vertex corrections contain already two Born-like contributions corresponding to modifications of one or the other vertex in the Born diagram (see Fig. 1). We can see the modified the vertex corrections (VC) corresponding to the diagrams shown in Fig. 5 where the photon and muon vertices contain additional virtual photon exchanges.

The diagrams in Fig. 4 correspond with the following amplitudes:

$$\begin{aligned} i\mathcal{M}_{1VC} &= -\bar{u}(k_1, m_\mu)(-ie\gamma_\alpha)\Gamma^a \frac{i(\hat{p}_2 - \hat{k}_2 + m_\mu)}{(k_2 - p_2)^2 - m_\mu^2} \\ &\quad \times (-ie\gamma_\beta)\Gamma^a v(k_2, m_\mu)\varepsilon_\alpha(p_1)\varepsilon_\beta(p_2); \\ i\mathcal{M}_{2VC} &= -\bar{u}(k_1, m_\mu)(-ie\gamma_\beta)\Gamma^a \frac{i(\hat{k}_1 - \hat{p}_2 + m_\mu)}{(p_2 - k_1)^2 - m_\mu^2} \\ &\quad \times (-ie\gamma_\alpha)\Gamma^a v(k_2, m_\mu)\varepsilon_\alpha(p_1)\varepsilon_\beta(p_2). \end{aligned} \quad (15)$$

Here, Γ^a is the photon-muon vertex function (form factor).

The vertex for the interaction of fermion f with a photon a is associated with the expression

$$ie\gamma_\mu\Gamma_f^a, \text{ where } \Gamma_f^a = v_f^a - a_f^a\gamma_5. \quad (16)$$

We can use the calculation [130], which the authors obtained the renormalized vertex form factors as the effective coupling constants

$$v_\mu^\gamma \rightarrow v_\mu^{F_\gamma}, \quad a_\mu^\gamma \rightarrow v_\mu^{F_\gamma}, \quad (17)$$

for the photon,

$$v_\mu^{F_\gamma} = \frac{\alpha}{4\pi} \cdot v_\mu^\gamma \Lambda_\mu^\gamma. \quad (18)$$

The vector and axial-vector couplings of fermion f to a photon are taken in the form

$$v_f^\gamma = -Q_f, \quad a_f^\gamma = 0. \quad (19)$$

The function Λ_μ^γ describes the contribution of the triangle vertex diagram involving an extra photon exchange as in in

[130]. It should be noted that the real part of this function contains a collinear logarithm and, in the ultrarelativistic approximation, assumes the following form in the t channel:

$$\begin{aligned} \text{Re } \Lambda_\mu^\gamma &= -2 \ln \frac{-t}{\lambda^2} \left(\ln \frac{-t}{m_\mu^2} - 1 \right) \\ &\quad + \ln \frac{-t}{m_\mu^2} + \ln^2 \frac{-t}{m_\mu^2} + \frac{\pi^2}{3} - 4. \end{aligned} \quad (20)$$

The contribution of a soft photon (with the energy ω , such that $\lambda < \omega < \Delta E$, where $\Delta E \ll \sqrt{s}$ is the softness threshold) is usually calculated. The cancellation of infrared contributions is the sum of virtual (including Boxes) and soft corrections (at $\Delta E \ll \sqrt{s}$).

In all numerical estimations performed in this study, the photon mass was fixed: $\lambda = 10^{-5}\sqrt{s}$.

It is seen that the boxes with photons suffer from infrared divergence, which is regularized by the fictitious photon mass λ . The cancellation of this non-physical quantity occurs in the sum with the contribution of the radiation of a real photon (see Fig. 6).

The contribution of a soft photon is calculated as usual (with an energy ω , such that $\lambda < \omega < \Delta E$, where $\Delta E \ll \sqrt{s}$ is the softness threshold). In numerical calculations, λ is canceled.

In order to verify analytically whether the total result is independent of λ , it is necessary to take the sum of expressions (11), (12), and the cross section of (20), and expression (23), and the cross section of (25).

Therefore, in order to have a physical meaning it is necessary to take into account all the contributions of radiation correction.

2.2 Soft photon emission

For infrared divergences cancellation in a Box diagram with photons, regularization with a fictitious photon with mass λ is applied.

The soft real photon emission process can be written in the following form

$$\gamma(p_1) + \gamma(p_2) \rightarrow \mu^-(k_1) + \mu^+(k_2) + \gamma(k). \quad (21)$$

The Feynman diagrams process (21) is shown in Fig. 6. The abbreviation of this non-physical quantity occurs in the sum with the contribution of the radiation of a real photon. Usually is calculated the contribution of a soft photon (with energy ω , such that $\lambda < \omega < \Delta E \ll \sqrt{s}$ - softness threshold) then the matrix element is factorized:

$$M^{SPE} \sim M_{Born} \cdot e_{\alpha}^* \left(\frac{k_1^{\alpha}}{(kk_1)} - \frac{k_2^{\alpha}}{(kk_2)} \right). \quad (22)$$

The cross section with radiative real photon can be written as follows:

$$\frac{d\sigma^{SPE}}{d\cos\theta} = \delta_{SPE} \cdot \frac{d\sigma^{Born}}{d\cos\theta}. \quad (23)$$

After performing integrations by the known methods of 't Hooft and Veltman [131], as a result for δ_{SPE} in the ultrarelativistic approximation we obtain following expression:

$$\begin{aligned} \delta_{SPE} &= -\frac{\alpha}{(2\pi)^2} \int_{\lambda < \omega < \Delta E} \left(\frac{k_1}{(kk_1)} - \frac{k_2}{(kk_2)} \right)^2 \cdot \frac{d^3\mathbf{k}}{\omega} \\ &= \frac{\alpha}{\pi} \left(2 \ln \frac{2\Delta E}{\lambda} \left(\ln \frac{st}{m_{\mu}^2 u} - 1 \right) + L_t \right. \\ &\quad \left. - \frac{L_t^2}{2} - \frac{\pi^2}{3} - Li_2 \left(-\frac{t}{u} \right) - Li_2 \left(-\frac{u}{t} \right) \right), \quad (24) \end{aligned}$$

where $L_t = \ln(-t/m_{\mu}^2)$.

It should be noted that the soft photon emission is evaluated in the soft regime where the energy of the photon in the centre-of-mass ω , is constrained by $\lambda < \omega < \Delta E \ll E = \frac{\sqrt{s}}{2}$. In numerical calculations we applied $\Delta E/E = 0.01$.

It should be noted that the infrared divergent part of the Box cross sections is frequently combined with the infrared divergent vertex part and the soft photon contribution, so that the parameter λ is canceled analytically.

But, we present a different procedure, which is considered complete formulas for the box cross sections, thereby retaining λ in the code for numerical estimations. Therefore, owing to this, the respective expressions remain complete and quite symmetric, moreover, the λ independence demonstrated numerically strengthens confidence in the correctness of programming.

Yes, a real photon its mass is zero. But, computation is performed with a fictitious photon with mass λ . Therefore, $\omega = 0$ cannot be taken, because in this case the expression will diverge. In such cases the minimum value of ω must be the λ parameter. The proper value of λ in equation (24) will be $\lambda = 10^{-5}\sqrt{s}$.

For $\sqrt{s}$ in the corresponding experiment L3 LEP it is possible to take 3 GeV or 40 GeV, but, $\sqrt{s}$ in all calculations for the value of parameter λ should be the same.

The applicability of the kinematically upper limit ($\Delta E/E = 0.01$) is sufficient.

2.3 Vacuum polarization

The total one loop virtual corrections include the decomposition of the virtual corrections amplitudes for example, the photon self energy (or vacuum polarization). Figure 7 corresponds to the diagram of the photon self-energy contribution.

The corresponding matrix elements for the diagrams in Fig. 7, can be written in the following form:

$$\begin{aligned} i\mathcal{M}_{1VP} &= -\bar{u}(k_1, m_{\mu})(-ie\gamma_{\alpha}) \cdot (-i\Pi(q)) \\ &\quad \cdot (-ie\gamma_{\beta})v(k_2, m_{\mu})\varepsilon_{\alpha}(p_1)\varepsilon_{\beta}(p_2); \\ i\mathcal{M}_{2VP} &= -\bar{u}(k_1, m_{\mu})(-ie\gamma_{\beta}) \cdot (-i\Pi(q)) \\ &\quad \cdot (-ie\gamma_{\alpha})v(k_2, m_{\mu})\varepsilon_{\alpha}(p_1)\varepsilon_{\beta}(p_2). \quad (25) \end{aligned}$$

In the first approximation, the mass operator is represented by a loop in the diagram (Fig. 7), and the vacuum polarization operator $\Pi(q)$ has the following form:

$$-i\Pi(q) = -\frac{ie^2}{(2\pi)^4} \int \frac{\gamma_{\mu}(\hat{q} - \hat{k} + m_{\mu})\gamma_{\nu} \cdot \delta_{\mu\nu}}{(k^2 - \lambda^2)((q-k)^2 - m_{\mu}^2)} d^4k. \quad (26)$$

Let us consider the behavior of the mass operator at $q^2 \gg m_{\mu}^2$, then we obtain an expression for the mass operator with logarithmic accuracy [129]:

$$\Pi(q) \approx \frac{\alpha}{4\pi} \hat{q} \ln \frac{q^2}{m_{\mu}^2}. \quad (27)$$

3 Transverse momentum, rapidity and angular distributions

In this section, we present the transverse-momentum, rapidity and angular distributions for the muon-pair production in photon-photon collisions.

It is well known that particles (e.g. leptons) produced in the final state should possess some intrinsic (primordial) transverse momentum. It should be noted that the recoil transverse momenta of dilepton pairs produced in hadronic collisions have been uniquely calculated in QCD [132–135, 137–140]. Similarly, the recoil transverse momenta of muon pairs produced in photon collisions also can be uniquely calculated in QED.

It should also be noted that intrinsic (primordial) transverse momenta of particles in the initial state are almost negligible; therefore, the experimentally observed large transverse momenta of massive muons produced in photons collisions have to arise almost entirely from the dynamical recoil effects.

Therefore, the process $\gamma\gamma \rightarrow \mu^+\mu^-$ is a particularly interesting one that produces massive lepton pairs of the large transverse-momentum which can help to provide information about the Standard Model structure.

In this way, a precise measurement of the p_T spectrum also provides an important input to the background prediction looking for the processes beyond the Standard Model.

Now we will study the distribution of the transverse momenta and the rapidity of muons in the final state corresponding to the process (1). Therefore, the Mandelstam variables are related to the transverse momentum and rapidity. Then, the Mandelstam invariants of the $\gamma\gamma \rightarrow \mu^+\mu^-$ processes satisfy the following form:

$$\begin{aligned} t &= m_\mu^2 - \sqrt{s} \sqrt{p_T^2 + m_\mu^2} e^{-y}, \\ u &= m_\mu^2 - \sqrt{s} \sqrt{p_T^2 + m_\mu^2} e^y, \end{aligned} \quad (28)$$

where p_T and y are the transverse momentum and rapidity of the muon, respectively.

In the center-of-mass system, the transverse momentum μ^- and μ^+ is obviously just opposite:

$$p_T \equiv p_T^{\mu^-} = -p_T^{\mu^+}. \quad (29)$$

The master formula for the differential cross section on the transverse momentum distribution has the form:

$$\frac{d\sigma}{dp_T^2} = \int dy \cdot \frac{d\sigma}{dp_T^2 dy}. \quad (30)$$

The integration limits for the transverse momentum are determined as follows:

$$0 \leq p_T \leq \frac{1}{2} \sqrt{s - 4m_\mu^2}. \quad (31)$$

Hence we can see that $p_T^{\max} = \frac{1}{2} \sqrt{s - 4m_\mu^2}$.

The differential cross section for the rapidity distribution can be written in the following form:

$$\frac{d\sigma}{dy} = \int dp_T^2 \cdot \frac{d\sigma}{dy dp_T^2}. \quad (32)$$

The kinematic limits of the rapidity in (30) determined by these relations, are given by:

$$-\ln \frac{2\sqrt{s}}{\sqrt{s - 4m_\mu^2}} \leq y \leq \ln \frac{2\sqrt{s}}{\sqrt{s - 4m_\mu^2}}. \quad (33)$$

In the most common case of equal masses in the final state, the differential cross section for the angular distribution has the following form:

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{32\pi s} \beta_\mu |\mathcal{M}|^2. \quad (34)$$

One of the interesting tasks in the study of the production of muon pairs in a photon–photon collision is to investigate the dependence of the cross-section on the muon transverse momentum. Therefore, we studied in detail the dependence of the differential cross-section on the muon transverse momentum in the $\gamma\gamma \rightarrow \mu^+\mu^-$ process (1).

In Figs. 8 and 9, we present our results of the investigation of the muon transverse momentum for different values of the centre-of-mass energy and rapidity for the process $\gamma\gamma \rightarrow \mu^+\mu^-$ (1).

In Fig. 8, we plot the dependence of the differential cross section for the process $\gamma\gamma \rightarrow \mu^+\mu^-$ (1) as a function of p_T in the range $0 < p_T < 10$ GeV/c for a fixed value of the centre-of-mass energy $\sqrt{s} = 20$ GeV and rapidity $y = -1, 0, 1, 2$.

We present the results for p_T in the range $0 \leq p_T \leq 15$ GeV/c at $\sqrt{s} = 30$ GeV and rapidity $y = -1, 0, 1, 2$ in Fig. 9.

The physical effects corresponding to Figs. 8 and 9 are given in the Appendix B.

Also, we investigate the dependence on the process $\gamma\gamma \rightarrow \mu^+\mu^-$ (1) as a function of the rapidity $y = y_{\mu^+} = y_{\mu^-}$ of the muon pairs in the range $-4 \leq y \leq 4$ for different values of the center-of-mass energy $\sqrt{s}$ and of the transverse momentum p_T .

One of the interesting problems, of the muon pair production in photon–photon collisions is the dependence of the angular distribution of the differential cross-section. Therefore, we studied the dependence of the differential cross-section on the muon angular distribution mean Eq. (34) in the $\gamma\gamma \rightarrow \mu^+\mu^-$ (1) process. In Fig. 11, we show the dependence of the differential cross section corresponding to the process $\gamma\gamma \rightarrow \mu^+\mu^-$ (1) as a function of $\cos\theta_\mu$ in the ranges $(-1 \leq \cos\theta_\mu \leq 1)$ and $(-0.8 \leq \cos\theta_\mu \leq 0.8)$ for two values of the centre-of-mass energy $\sqrt{s}$.

4 The process $e^+ + e^- \rightarrow e^+ + e^- + \mu^+ + \mu^-$

In the interaction of an electron and positron many processes occur being one of them the creation of $\mu^+\mu^-$ pairs.

One of the most important areas of QED in the context of SM is the investigation of the production of massive lepton pairs at the electron–positron linear collider.

In this paper, we report the results from a study of the QED process $e^+e^- \rightarrow e^+e^-\mu^+\mu^-$ up to the α^4 order- at the center-of mass with energies from 161 GeV to 209 GeV.

The lepton pair production in electron–positron collisions by a two photon mechanism has been studied for a long time and plays a special role in QED and QCD [19], since their analysis is under much better control.

The process of production of lepton pairs in electron–positron collisions via photon–photon-fusion can be written in the form:

$$e^+ + e^- \rightarrow e^+ + \gamma^* + e^- + \gamma^* \rightarrow e^+ + e^- + \mu^+ + \mu^-. \quad (35)$$

The Feynman diagrams of the process (35) are represented in Fig. 12.

The total cross section of the $e^+e^- \rightarrow e^+e^-\mu^+\mu^-$ process, which we study through photon-photon-fusion subprocess can be written as [19,24]:

$$\sigma(s)^{e^+e^- \rightarrow e^+e^-\mu^+\mu^-} = \left(\frac{\alpha}{\pi}\right)^2 \int_{s_1^{\min}}^{s_1^{\max}} \frac{ds_1}{s_1} \sigma^{\gamma\gamma \rightarrow \mu^+\mu^-}(s_1) \times \left[\left(\ln \frac{s^2}{4s_1 m_e^2} - 1 \right)^2 f\left(\frac{s}{s_1}\right) - \frac{1}{3} \left(\ln \frac{s}{s_1} \right)^3 \right], \quad (36)$$

where

$$f(x) = \left(1 + \frac{1}{2x}\right)^2 \ln x - \frac{1}{2} \left(1 - \frac{1}{x}\right) \left(3 + \frac{1}{x}\right), \quad (37)$$

where $\sigma^{\gamma\gamma \rightarrow \mu^+\mu^-}(s_1)$ is the cross section of the $\gamma\gamma \rightarrow \mu^+\mu^-$ process.

Physical allowed region for the $e^+ + e^- \rightarrow e^+e^-\mu^+\mu^-$ process in accordance with experiment L3 LEP [92] is: $3 \text{ GeV} \leq W_{\gamma\gamma} \leq 40 \text{ GeV}$, $\sqrt{s_1} = W_{\gamma\gamma}$. Integration over s_1 is performed in region $9 \text{ GeV}^2 \leq s_1 \leq 1600 \text{ GeV}^2$.

Using the two-photon mechanism, we calculated the total cross section of the $e^+ + e^- \rightarrow e^+ + e^- + \mu^+ + \mu^-$ process (35), according to formula (36), in a large range of centre-of-mass energies $\sqrt{s}$ and for energies that correspond to the experiment with a specific angle range. In Fig. 13, the dependence of the total cross section on the center-of-mass energy of the $e^+ + e^- \rightarrow e^+ + e^- + \mu^+ + \mu^-$ process is plotted.

We also investigated the dependence of the cross section as a function of $\sqrt{s}$ in the $e^+ + e^- \rightarrow e^+ + e^- + \mu^+ + \mu^-$ (35) process for the case of two-photons with the energy in the center-of-mass in the range of $3 \text{ GeV} \leq W_{\gamma\gamma} \leq 40 \text{ GeV}$ and muon angles in the range $|\cos\theta_\mu| \leq 0.8$. Therefore, in formula (36) for the $\gamma\gamma \rightarrow \mu^+\mu^-$ subprocess we use formula (34) and perform the integration over $W_{\gamma\gamma}$ in the range $3 \text{ GeV} \leq W_{\gamma\gamma} \leq 40 \text{ GeV}$ and over the muon angle in the region $(-0.8 \leq \cos\theta_\mu \leq 0.8)$.

5 Conclusion

As it is well known, the photon-photon and electron-positron collisions are basically the most elementary interactions and form the basis of our knowledge about the nature of high energy physics.

The exclusive production of $\mu^+\mu^-$ pairs in electron-positron collisions via the two-photon mechanism is considered. In this process, we present a simple method for calculating the total and differential cross section.

A careful study of the muon pair production reactions in LEP, LHC, BABAR, BELLE, Fermilab et al. over the full range of $\sqrt{s}$ values and high p_T is highly desirable, both: as an important probe and as a significant test of the SM.

After a detailed study of the photon-photon-collision processes, one can notice that two-photon reactions have unique features to test QED and QCD.

To compute the dependence of the total and differential cross section on the centre-of-mass energy, transverse momentum, and the rapidity angle, we use the master formula for the $\gamma\gamma \rightarrow \mu^+\mu^-$ process.

It should be noted that the total cross section of the $e^+e^- \rightarrow e^+e^-\mu^+\mu^-$ process with two-photon, the center-of-mass energy was measured in the region of $3 \text{ GeV} \leq W_{\gamma\gamma} \leq 40 \text{ GeV}$ by using the data taken from the L3 detector at $161 \text{ GeV} \leq \sqrt{s} \leq 209 \text{ GeV}$ at the LEP.

We also want to pointed out that the total cross section of the muon pair production as a function of the center-of-mass energy of the $\gamma\gamma \rightarrow \mu^+\mu^-$ process was measured from the two-photon center-of-mass energy $\sqrt{s}$ in the range $3 \text{ GeV} \leq W_{\gamma\gamma} \leq 40 \text{ GeV}$ using the data, from the L3 detector at the LEP.

In this paper, we have examined the muon pair production in photon - photon collisions and in the process $e^+e^- \rightarrow e^+e^-\mu^+\mu^-$ by mean the two-photon mechanism $\gamma\gamma \rightarrow \mu^+\mu^-$.

Also, we investigated in detail the dependence of the total cross section as a function of the two-photon centre-of-mass energy in the range of $3 \text{ GeV} \leq W_{\gamma\gamma} \leq 40 \text{ GeV}$, and the dependence of the differential cross section on the transverse momentum p_T in the region $2 \text{ GeV}/c \leq p_T \leq 200 \text{ GeV}/c$ for different values of the centre-of-mass energy $\sqrt{s}=10, 20 \text{ GeV}$, and of rapidity $y = -1, 0, 1, 2$, in the process $\gamma\gamma \rightarrow \mu^+\mu^-$.

We also investigated the dependence of the differential cross section on rapidity y for different values of centre-of-mass energy $\sqrt{s} = 10, 20 \text{ GeV}$, and of transverse momentum $p_T = 5, 10 \text{ GeV}/c$ in the process $\gamma\gamma \rightarrow \mu^+\mu^-$.

The detail of the angular distribution of the differential cross section $\frac{d\sigma(\gamma\gamma \rightarrow \mu^+\mu^-)}{d\cos\theta_\mu}$ of the $\gamma\gamma \rightarrow \mu^+\mu^-$ process is also studied where we considered separately the angle regions in the range of $(-1 \leq \cos\theta_\mu \leq 1)$ and $(-0.8 \leq \cos\theta_\mu \leq 0.8)$, and for different values of the centre-of-mass energy of the two-photon system $\sqrt{s}=10, 20 \text{ GeV}$.

Finally, the $\mu^+\mu^-$ production in the $e^+ + e^- \rightarrow e^+ + e^- + \mu^+ + \mu^-$ process is investigated, studying the characteristics of the total cross section as a function of the e^+e^- center-of-mass energy $\sqrt{s}$ in the region $5 \text{ GeV} \leq \sqrt{s} \leq 209 \text{ GeV}$ of this process for different values of angle regions $-1 \leq \cos\theta_\mu \leq 1$ and $-0.8 \leq \cos\theta_\mu \leq 0.8$ in the two foton mechanism.

With respect to the figures, in Fig. 9d we show a comparison of the results of the Born-level diagram with respect

to the Born + Box + Vertex correction + Soft photon emission + Vacuum polarization diagrams. It is shown that the contribution from the Box + Vertex correction + Soft photon emission + Vacuum polarization diagrams is about 16–18%.

In Fig. 10b, we show a comparison of the results of the Born-level diagram with respect to the Born + Box + Vertex correction + Soft photon emission + Vacuum polarization diagrams. The contribution from the Box + Vertex correction + Soft photon emission + Vacuum polarization diagrams is about 16–18%.

In Fig. 11b, c, we show a comparison of the results of the Born-level diagram with respect to the Born + Box + Vertex correction + Soft photon emission + Vacuum polarization diagrams. It is shown that the contribution from the Box + Vertex correction + Soft photon emission + Vacuum polarization diagrams is about 15–18%, and 16–19%, respectively.

We want to note that some of our obtained theoretical results are compared with experimental data for the processes of $\gamma\gamma \rightarrow \mu^+\mu^-$ and $e^+e^- \rightarrow e^+e^- + \mu^+ + \mu^-$. They are in satisfactory agreement.

To evaluate the loop integrals in the Box diagrams, we used the packages LoopTools [141] based on the Passarino-Veltman reduction techniques [142].

In our opinion, our theoretical results will be helpful for investigations and analysis of the lepton pair production at e^+e^- and the other colliders.

6 Appendix A

After integration in (6) by t , according to formula (9), for the square of the matrix element of the Born diagram, we obtain the following expression:

$$\begin{aligned} |M_{\text{Born}}|^2 = & (4\pi\alpha)^2 \left\{ 2sm_\mu^2 - 10m_\mu^4 \right\} \cdot \frac{\sqrt{s(s-4m_\mu^2)}}{sm_\mu^2} \\ & - 2s \cdot \left(\frac{\sqrt{s(s-4m_\mu^2)}}{s} + \ln \frac{s - \sqrt{s(s-4m_\mu^2)}}{s + \sqrt{s(s-4m_\mu^2)}} \right) \\ & - \frac{2}{s} (s + m_\mu^2) \sqrt{s(s-4m_\mu^2)} \\ & + \left(6sm_\mu^2 - 18m_\mu^4 \right) \cdot \frac{\sqrt{s(s-4m_\mu^2)}}{sm_\mu^2} \\ & + (8m_\mu^2 - 2s) \cdot \left(\frac{m_\mu^2 - s}{m_\mu^2 s} \sqrt{s(s-4m_\mu^2)} \right. \\ & \left. + \ln \frac{s + \sqrt{s(s-4m_\mu^2)}}{s - \sqrt{s(s-4m_\mu^2)}} \right) \\ & - 2 \left(\frac{1}{sm_\mu^2} (m_\mu^4 - sm_\mu^2 + s^2) \sqrt{s(s-4m_\mu^2)} \right. \end{aligned}$$

$$\begin{aligned} & \left. + 2(m_\mu^2 - s) \ln \frac{s + \sqrt{s(s-4m_\mu^2)}}{s - \sqrt{s(s-4m_\mu^2)}} \right) \\ & + \frac{1}{s} \left(8sm_\mu^2 - 12m_\mu^4 \right) \left(\ln \frac{s + \sqrt{s(s-4m_\mu^2)}}{s - \sqrt{s(s-4m_\mu^2)}} \right. \\ & \left. + \ln \frac{-s - \sqrt{s(s-4m_\mu^2)}}{-s + \sqrt{s(s-4m_\mu^2)}} \right) \}. \quad (38) \end{aligned}$$

The terms $M_{\text{Born}}^+ \cdot M_{\text{Box}}$ can be written in the following form:

$$\begin{aligned} \mathcal{M}_1^+ \cdot \mathcal{M}_3 = & -\frac{(4\pi\alpha)^3}{\pi^4} \frac{g_{\mu\nu}g_{\alpha\alpha'}g_{\beta\beta'}}{(k_2 - p_2)^2 - m_\mu^2} \\ & \times \int d^4q_1 \frac{\text{Tr}(\hat{k}_2 - m_\mu)\gamma_{\beta'}(\hat{p}_2 - \hat{k}_2 + m_\mu)\gamma_{\alpha'}(\hat{k}_1 + m_\mu)\gamma_\mu}{(q_1^2 - m_\mu^2)((p_2 + q_1)^2 - m_\mu^2)} \\ & \cdot \frac{(-\hat{p}_2 - \hat{q}_1 + \hat{k}_1 + \hat{k}_2 + m_\mu)\gamma_\alpha(-\hat{q}_1 + m_\mu)\gamma_\beta(-\hat{p}_2 - \hat{q}_1 + m_\mu)\gamma_\nu}{(p_2 + q_1 - k_2)^2((p_2 + q_1 - k_1 - k_2)^2 - m_\mu^2)}, \\ \mathcal{M}_1^+ \cdot \mathcal{M}_4 = & -\frac{16G_F^3 \sin^6\theta_W M_W^6}{\pi^4} \\ & \times \left(\frac{\sin^2\theta_W - \frac{1}{2}}{\cos\theta_W \sin\theta_W} + \frac{\sin\theta_W}{\cos\theta_W} \right)^2 \frac{g_{\mu\nu}g_{\alpha\alpha'}g_{\beta\beta'}}{(k_2 - p_2)^2 - m_\mu^2} \\ & \cdot \int d^4q_1 \frac{\text{Tr}(\hat{k}_2 - m_\mu)\gamma_{\beta'}(\hat{p}_2 - \hat{k}_2 + m_\mu)\gamma_{\alpha'}(\hat{k}_1 + m_\mu)\gamma_\mu}{(q_1^2 - m_\mu^2)((p_2 + q_1)^2 - m_\mu^2)((p_2 + q_1 - k_2)^2 - M_W^2)} \\ & \cdot \frac{(-\hat{p}_2 - \hat{q}_1 + \hat{k}_1 + \hat{k}_2 + m_\mu)\gamma_\alpha(-\hat{q}_1 + m_\mu)\gamma_\beta(-\hat{p}_2 - \hat{q}_1 + m_\mu)\gamma_\nu}{((p_2 + q_1 - k_1 - k_2)^2 - m_\mu^2)}, \\ \mathcal{M}_1^+ \cdot \mathcal{M}_5 = & \frac{(4\pi\alpha)^2 M_W^2 G_F}{4\sqrt{2}\pi^4} \frac{g_{\mu\nu}g_{\rho\sigma}g_{\lambda\kappa}g_{\alpha\alpha'}g_{\beta\beta'}}{(k_2 - p_2)^2 - m_\mu^2} \\ & \cdot \int d^4q_1 \frac{\text{Tr}(\hat{k}_2 - m_\mu)\gamma_{\beta'}(\hat{p}_2 - \hat{k}_2 + m_\mu)\gamma_{\alpha'}}{(q_1^2 - M_W^2)((p_2 + q_1)^2 - M_W^2)} \\ & \cdot \frac{(\hat{k}_1 + m_\mu)\gamma_\lambda(-\hat{p}_2 - \hat{q}_1 + \hat{k}_1)\gamma_\sigma}{((p_2 + q_1 - k_1 - k_2)^2 - M_W^2)((p_2 + q_1 - k_2)^2 - M_W^2)} \\ & \cdot \left((-p_1 - q_1)^\rho g_{\alpha\mu} + (p_1 - p_2 - q_1 + k_1 + k_2)^\mu g_{\alpha\rho} \right. \\ & \left. + (p_2 + 2q_1 - k_1 - k_2)^\alpha g_{\mu\rho} \right) \\ & \cdot \left((-p_2 + q_1)^\kappa g_{\beta\nu} + (2p_2 + q_1)^\nu g_{\beta\kappa} + (-p_2 - 2q_1)^\beta g_{\nu\kappa} \right), \quad (39) \end{aligned}$$

$$\begin{aligned} \mathcal{M}_1^+ \cdot \mathcal{M}_6 = & \frac{(4\pi\alpha)^3}{\pi^4} \frac{g_{\mu\nu}g_{\alpha\alpha'}g_{\beta\beta'}}{(k_2 - p_2)^2 - m_\mu^2} \\ & \times \int d^4q_1 \frac{\text{Tr}(\hat{k}_2 - m_\mu)\gamma_{\beta'}(\hat{p}_2 - \hat{k}_2 + m_\mu)\gamma_{\alpha'}(\hat{k}_1 + m_\mu)\gamma_\mu}{(q_1^2 - m_\mu^2)((p_2 + q_1)^2 - m_\mu^2)} \\ & \cdot \frac{(\hat{p}_2 + \hat{q}_1 + m_\mu)\gamma_\beta(\hat{q}_1 + m_\mu)\gamma_\alpha(\hat{p}_2 + \hat{q}_1 - \hat{k}_1 - \hat{k}_2 + m_\mu)\gamma_\nu}{(p_2 + q_1 - k_1)^2((p_2 + q_1 - k_1 - k_2)^2 - m_\mu^2)}, \\ \mathcal{M}_1^+ \cdot \mathcal{M}_7 = & -\frac{16G_F^3 \sin^6\theta_W M_W^6}{\pi^4} \\ & \times \left(\frac{\sin^2\theta_W - \frac{1}{2}}{\cos\theta_W \sin\theta_W} + \frac{\sin\theta_W}{\cos\theta_W} \right)^2 \frac{g_{\mu\nu}g_{\alpha\alpha'}g_{\beta\beta'}}{(k_2 - p_2)^2 - m_\mu^2} \\ & \cdot \int d^4q_1 \frac{\text{Tr}(\hat{k}_2 - m_\mu)\gamma_{\beta'}(\hat{p}_2 - \hat{k}_2 + m_\mu)\gamma_{\alpha'}(\hat{k}_1 + m_\mu)\gamma_\mu}{(q_1^2 - m_\mu^2)((p_2 + q_1)^2 - m_\mu^2)((p_2 + q_1 - k_1)^2 - M_W^2)} \\ & \cdot \frac{(\hat{p}_2 + \hat{q}_1 + m_\mu)\gamma_\beta(\hat{q}_1 + m_\mu)\gamma_\alpha(\hat{p}_2 + \hat{q}_1 - \hat{k}_1 - \hat{k}_2 + m_\mu)\gamma_\nu}{((p_2 + q_1 - k_1 - k_2)^2 - m_\mu^2)}, \\ \mathcal{M}_1^+ \cdot \mathcal{M}_8 = & \frac{(4\pi\alpha)^2 M_W^2 G_F}{4\sqrt{2}\pi^4} \frac{g_{\mu\nu}g_{\rho\sigma}g_{\lambda\kappa}g_{\alpha\alpha'}g_{\beta\beta'}}{(k_2 - p_2)^2 - m_\mu^2} \\ & \cdot \int d^4q_1 \frac{\text{Tr}(\hat{k}_2 - m_\mu)\gamma_{\beta'}(\hat{p}_2 - \hat{k}_2 + m_\mu)\gamma_{\alpha'}}{(q_1^2 - M_W^2)((p_2 + q_1)^2 - M_W^2)} \end{aligned}$$

$$\begin{aligned}
& \cdot \frac{(\hat{k}_1 + m_\mu)\gamma_\lambda(-\hat{p}_2 - \hat{q}_1 + \hat{k}_1)\gamma_\sigma}{((p_2 + q_1 - k_1 - k_2)^2 - M_W^2)(p_2 + q_1 - k_1)^2} \\
& \cdot \left((-p_1 - q_1)^\rho g_{\alpha\mu} + (p_1 - p_2 - q_1 + k_1 + k_2)^\mu g_{\alpha\rho} + \right. \\
& \quad \left. + (p_2 + 2q_1 - k_1 - k_2)^\alpha g_{\mu\rho} \right) \\
& \cdot \left((-p_2 + q_1)^\kappa g_{\beta\nu} + (2p_2 + q_1)^\nu g_{\beta\kappa} + (-p_2 - 2q_1)^\beta g_{\nu\kappa} \right), \\
\mathcal{M}_2^+ \cdot \mathcal{M}_3 = & -\frac{(4\pi\alpha)^3}{\pi^4} \frac{g_{\mu\nu}g_{\alpha\alpha'}g_{\beta\beta'}}{(p_2 - k_1)^2 - m_\mu^2} \\
& \times \int d^4q_1 \frac{Tr(\hat{k}_2 - m_\mu)\gamma_{\alpha'}(\hat{k}_1 - \hat{p}_2 + m_\mu)\gamma_{\beta'}(\hat{k}_1 + m_\mu)\gamma_\mu}{(q_1^2 - m_\mu^2)((p_2 + q_1)^2 - m_\mu^2)} \\
& \cdot \frac{(-\hat{p}_2 - \hat{q}_1 + \hat{k}_1 + \hat{k}_2 + m_\mu)\gamma_\alpha(-\hat{q}_1 + m_\mu)\gamma_\beta(-\hat{p}_2 - \hat{q}_1 + m_\mu)\gamma_\nu}{(p_2 + q_1 - k_2)^2((p_2 + q_1 - k_1 - k_2)^2 - m_\mu^2)}, \\
\mathcal{M}_2^+ \cdot \mathcal{M}_4 = & -\frac{16G_F^3 \sin^6\theta_W M_W^6}{\pi^4} \\
& \times \left(\frac{\sin^2\theta_W - \frac{1}{2}}{\cos\theta_W \sin\theta_W} + \frac{\sin\theta_W}{\cos\theta_W} \right)^2 \frac{g_{\mu\nu}g_{\alpha\alpha'}g_{\beta\beta'}}{(p_2 - k_1)^2 - m_\mu^2} \\
& \cdot \int d^4q_1 \frac{Tr(\hat{k}_2 - m_\mu)\gamma_{\alpha'}(\hat{k}_1 - \hat{p}_2 + m_\mu)\gamma_{\beta'}(\hat{k}_1 + m_\mu)\gamma_\mu}{(q_1^2 - m_\mu^2)((p_2 + q_1)^2 - m_\mu^2)((p_2 + q_1 - k_2)^2 - M_Z^2)} \\
& \cdot \frac{(-\hat{p}_2 - \hat{q}_1 + \hat{k}_1 + \hat{k}_2 + m_\mu)\gamma_\alpha(-\hat{q}_1 + m_\mu)\gamma_\beta(-\hat{p}_2 - \hat{q}_1 + m_\mu)\gamma_\nu}{((p_2 + q_1 - k_1 - k_2)^2 - m_\mu^2)}, \\
\mathcal{M}_2^+ \cdot \mathcal{M}_5 = & \frac{(4\pi\alpha)^2 M_W^2 G_F}{4\sqrt{2}\pi^4} \frac{g_{\mu\nu}g_{\rho\sigma}g_{\lambda\kappa}g_{\alpha\alpha'}g_{\beta\beta'}}{(p_2 - k_1)^2 - m_\mu^2} \\
& \cdot \int d^4q_1 \frac{Tr(\hat{k}_2 - m_\mu)\gamma_{\alpha'}(\hat{k}_1 - \hat{p}_2 + m_\mu)\gamma_{\beta'}}{(q_1^2 - M_W^2)((p_2 + q_1)^2 - M_W^2)} \\
& \cdot \frac{(\hat{k}_1 + m_\mu)\gamma_\sigma(\hat{p}_2 + \hat{q}_1 - \hat{k}_2)\gamma_\kappa}{((p_2 + q_1 - k_1 - k_2)^2 - M_W^2)(p_2 + q_1 - k_2)^2} \\
& \cdot \left((-p_1 - q_1)^\rho g_{\alpha\mu} + (p_1 - p_2 - q_1 + k_1 + k_2)^\mu g_{\alpha\rho} \right. \\
& \quad \left. + (p_2 + 2q_1 - k_1 - k_2)^\alpha g_{\mu\rho} \right) \\
& \cdot \left((-p_2 + q_1)^\lambda g_{\beta\nu} + (2p_2 + q_1)^\nu g_{\beta\lambda} + (-p_2 - 2q_1)^\beta g_{\nu\lambda} \right), \\
\mathcal{M}_2^+ \cdot \mathcal{M}_6 = & \frac{(4\pi\alpha)^3}{\pi^4} \frac{g_{\mu\nu}g_{\alpha\alpha'}g_{\beta\beta'}}{(p_2 - k_1)^2 - m_\mu^2} \\
& \times \int d^4q_1 \frac{Tr(\hat{k}_2 - m_\mu)\gamma_{\alpha'}(\hat{k}_1 - \hat{p}_2 + m_\mu)\gamma_{\beta'}(\hat{k}_1 + m_\mu)\gamma_\mu}{(q_1^2 - m_\mu^2)((p_2 + q_1)^2 - m_\mu^2)} \\
& \cdot \frac{(\hat{p}_2 + \hat{q}_1 + m_\mu)\gamma_\beta(\hat{q}_1 + m_\mu)\gamma_\alpha(\hat{p}_2 + \hat{q}_1 - \hat{k}_1 - \hat{k}_2 + m_\mu)\gamma_\nu}{(p_2 + q_1 - k_1)^2((p_2 + q_1 - k_1 - k_2)^2 - m_\mu^2)}, \\
\mathcal{M}_2^+ \cdot \mathcal{M}_7 = & -\frac{16G_F^3 \sin^6\theta_W M_W^6}{\pi^4} \\
& \times \left(\frac{\sin^2\theta_W - \frac{1}{2}}{\cos\theta_W \sin\theta_W} + \frac{\sin\theta_W}{\cos\theta_W} \right)^2 \frac{g_{\mu\nu}g_{\alpha\alpha'}g_{\beta\beta'}}{(p_2 - k_1)^2 - m_\mu^2} \\
& \cdot \int d^4q_1 \frac{Tr(\hat{k}_2 - m_\mu)\gamma_{\alpha'}(\hat{k}_1 - \hat{p}_2 + m_\mu)\gamma_{\beta'}(\hat{k}_1 + m_\mu)\gamma_\mu}{(q_1^2 - m_\mu^2)((p_2 + q_1)^2 - m_\mu^2)((p_2 + q_1 - k_1)^2 - M_Z^2)} \\
& \cdot \frac{(\hat{p}_2 + \hat{q}_1 + m_\mu)\gamma_\beta(\hat{q}_1 + m_\mu)\gamma_\alpha(\hat{p}_2 + \hat{q}_1 - \hat{k}_1 - \hat{k}_2 + m_\mu)\gamma_\nu}{((p_2 + q_1 - k_1 - k_2)^2 - m_\mu^2)}, \\
\mathcal{M}_2^+ \cdot \mathcal{M}_8 = & \frac{(4\pi\alpha)^2 M_W^2 G_F}{4\sqrt{2}\pi^4} \frac{g_{\mu\nu}g_{\rho\sigma}g_{\lambda\kappa}g_{\alpha\alpha'}g_{\beta\beta'}}{(p_2 - k_1)^2 - m_\mu^2} \\
& \cdot \int d^4q_1 \frac{Tr(\hat{k}_2 - m_\mu)\gamma_{\alpha'}(\hat{k}_1 - \hat{p}_2 + m_\mu)\gamma_{\beta'}}{(q_1^2 - M_W^2)((p_2 + q_1)^2 - M_W^2)} \\
& \cdot \frac{(\hat{k}_1 + m_\mu)\gamma_\lambda(-\hat{p}_2 - \hat{q}_1 + \hat{k}_1)\gamma_\sigma}{((p_2 + q_1 - k_1 - k_2)^2 - M_W^2)(p_2 + q_1 - k_1)^2} \\
& \cdot \left((-p_1 - q_1)^\rho g_{\alpha\mu} + (p_1 - p_2 - q_1 + k_1 + k_2)^\mu g_{\alpha\rho} \right.
\end{aligned}$$

$$\begin{aligned}
& \left. + (p_2 + 2q_1 - k_1 - k_2)^\alpha g_{\mu\rho} \right) \\
& \cdot \left((-p_2 + q_1)^\kappa g_{\beta\nu} + (2p_2 + q_1)^\nu g_{\beta\kappa} + (-p_2 - 2q_1)^\beta g_{\nu\kappa} \right),
\end{aligned}$$

where the quantities M_W , M_Z , m_μ , $e = \sqrt{4\pi\alpha}$, G_F , and θ_W are the W-boson mass, the Z-boson mass, the muon mass, the elementary electric charge ($\alpha \sim 1/137$), the Fermi coupling constant, and the weak mixing angle, respectively.

7 Appendix B

Additional details of the plots and figures below.

In Fig. 8a, at values of rapidity $y = 0$, and of p_T from 0 to 3 GeV/c, the differential cross section decreases very much and then in the range $3 \text{ GeV/c} \leq p_T \leq 10 \text{ GeV/c}$ the differential cross section decreases smoothly.

On the other curves in Fig. 8, that is, in Fig. 8b, c, we plot the differential cross section as a function of the muon transverse momentum in the range $0 \leq p_T \leq 10 \text{ GeV/c}$ at rapidity $y = 1$ and $y = -1$. At a value of p_T from 0 GeV/c to 4 GeV/c the differential cross section decreases very much and then in the range $4 \text{ GeV/c} \leq p_T \leq 10 \text{ GeV/c}$ the differential cross section decreases smoothly.

Also, we plot in Fig. 8d the transverse momentum distribution in the range $0 \leq p_T \leq 10 \text{ GeV/c}$ at values of rapidity $y = 2$ and $\sqrt{s} = 20 \text{ GeV}$. It should be noted that at a value of p_T from 0 to 3.5 GeV/c the differential cross section decreases very much, and then in the region $3.5 \text{ GeV/c} \leq p_T \leq 6 \text{ GeV/c}$ the differential cross section remains almost stable. With increasing in p_T up to $p_T = 10 \text{ GeV/c}$, the differential cross section increases very slowly.

In Fig. 9a at values of rapidity $y = 0$, and of p_T from 0 to 4 GeV/c the differential cross section decreases very much and then in the range $4 \text{ GeV/c} \leq p_T \leq 15 \text{ GeV/c}$ the differential cross section decreases smoothly.

On other curves in Fig. 9, that is, in Fig. 9b, c, we plot the differential cross section as a function of the muon transverse momentum in the range $0 \leq p_T \leq 15 \text{ GeV/c}$ at rapidity $y = 1$ and $y = -1$. At a value of p_T from 0 to 5.5 GeV/c the differential cross section decreases very much, and further with increasing p_T in region $5.5 \text{ GeV/c} \leq p_T \leq 15 \text{ GeV/c}$ the cross section decreases slowly.

Also, we plotted in Fig. 9d the transverse momentum distribution in the range $0 \leq p_T \leq 15 \text{ GeV/c}$ with values of rapidity $y = 2$ and $\sqrt{s} = 30 \text{ GeV}$. It should be noted that at a value of p_T from 0 to 5.5 GeV/c the differential cross section decreases very much, in the region $5.5 \text{ GeV/c} \leq p_T \leq 10 \text{ GeV/c}$ the differential cross section remains almost stable. In the region $10 \text{ GeV/c} \leq p_T \leq 15 \text{ GeV/c}$ the differential cross section increases slowly. In Fig. 9d we show a comparison of the results of the Born-level diagram with the Born+Box+Vertex correction+ Soft photon emission +

Fig. 8 Transverse momentum distribution of $\mu^\pm$ for a fixed value of the centre-of-mass energy $\sqrt{s}$ and for different values of y rapidity. **a** $\sqrt{s} = 20$ GeV, $y = 0$; **b** $\sqrt{s} = 20$ GeV, $y = 1$; **c** $\sqrt{s} = 20$ GeV, $y = -1$; **d** $\sqrt{s} = 20$ GeV, $y = 2$

Vacuum polarization diagrams. It is shown that the contribution from the Box+Vertex correction+Soft photon emission+Vacuum polarization diagrams is about 16–18%.

In Fig. 10a, we have depicted the differential cross section as a function of the muon rapidity y at $\sqrt{s} = 10$ GeV and $p_T = 5$ GeV/c. In the region $(-4 \leq y \leq -2)$ the differential cross section decreases sharply, but in the region $(-2 \leq y \leq 2)$ it changes very slowly (it is practically stable). We can see that we have a minimum value of the differential cross section at the point $y = 0$. Further, in the region $(2 \leq y \leq 4)$ the differential cross section increases sharply.

Analogous calculations have been carried out for $\sqrt{s} = 20$ GeV and $p_T = 10$ GeV/c. The obtained results are presented in Fig. 10b. Must be noted that for these values of $\sqrt{s} = 20$ GeV and $p_T = 10$ GeV/c, the same, in the region $(-4 \leq y \leq -2)$ the differential cross section decreases sharply, but in the region $(-2 \leq y \leq 2)$ it changes very slowly (is practically stable). It is worth mentioning that at $y = 0$ the differential cross section obtains a minimum value. In the region $(2 \leq y \leq 4)$ the differential cross section increases sharply. In Fig. 10b, we

show a comparison of the results of the Born-level diagram with the Born+Box+Vertex correction+ Soft photon emission+Vacuum polarization diagrams. The contribution from the Box+Vertex correction+Soft photon emission+Vacuum polarization diagrams is about 16–18%.

In Fig. 11a we plotted the differential cross section as a function of the muon angular distribution in the region $(-1 \leq \cos \theta_\mu \leq 1)$ at $\sqrt{s} = 10$ GeV. In the region $(-1 \leq \cos \theta_\mu \leq -0.2)$ the differential cross section decreases sharply, but in the region $(-0.2 \leq \cos \theta_\mu \leq 0.2)$ it changes very slowly. At $\theta_\mu = 90^\circ$ the differential cross section obtains a minimum value. It is seen that further in the region $(0.2 \leq \cos \theta_\mu \leq 1)$ the differential cross section increases sharply.

Analogous calculations have been carried out for $\sqrt{s} = 20$ GeV, and this dependence is displayed in Fig. 11b. It is shown that in the region $(-1 \leq \cos \theta_\mu \leq -0.2)$ the differential cross section decreases sharply, but in the region $(-0.2 \leq \cos \theta_\mu \leq 0.2)$ it changes very slowly. At $\theta_\mu = 90^\circ$ the differential cross section obtains a minimum value. In the region $(0.2 \leq \cos \theta_\mu \leq 1)$ the differential cross section increases sharply.

Fig. 9 Transverse momentum distribution of $\mu^\pm$ for a fixed value of the centre-of-mass energy $\sqrt{s}$ and for different values of rapidity y . **a** $\sqrt{s} = 30$ GeV, $y = 0$; **b** $\sqrt{s} = 30$ GeV, $y = 1$; **c** $\sqrt{s} = 30$ GeV,

$y = -1$; **d** $\sqrt{s} = 30$ GeV, $y = 2$; direct line – Born + Box + VC + SPE + VP diagrams, dashed line – Born-level process

Fig. 10 The dependence of the differential cross section on rapidity y for different values of the centre-of-mass energy $\sqrt{s}$ and transverse momentum p_T . **a** $\sqrt{s} = 10$ GeV, $p_T = 5$ GeV/c; **b** $\sqrt{s} = 20$ GeV, $p_T = 10$ GeV/c; direct line – Born+Box+VC+VP+SPE diagrams, dashed line – Born-level process

Fig. 11 The angular distribution of the differential cross section of the process $\gamma\gamma \rightarrow \mu^+\mu^-$ for different values of the centre-of-mass energy $\sqrt{s}$ and angle $\cos\theta_\mu$; direct line – Born+Box+VC+SPE+VP diagrams, dashed line – Born-level process

Fig. 12 Representative Feynman diagrams for the process of $e^+ + e^- \rightarrow e^+ + e^- + \mu^+ + \mu^-$

It should be noted that in the experiment the angular distribution of the differential cross section was also measured in the region $|\cos\theta_\mu| \leq 0.8$. Therefore, we have made analogous calculations for the angular distribution of the differential cross section in the region $(-0.8 \leq \cos\theta_\mu \leq 0.8)$ at $\sqrt{s} = 20$ GeV. The obtained results of our numerical calculation are plotted in Fig. 11c. It is shown that in the region $(-0.8 \leq \cos\theta_\mu \leq -0.1)$ the differential cross section decreases sharply, but in the region $(-0.1 \leq \cos\theta_\mu \leq 0.1)$ the differential cross section changes very slowly. At the value $\theta_\mu = 90^\circ$ the dif-

ferential cross section obtains a minimum value. Farther, in the region $(0.1 \leq \cos\theta_\mu \leq 0.8)$ the differential cross section increases sharply. In Fig. 11b, c, we show a comparison of the results of the Born-level diagram with the Born+Box+Vertex correction+Soft photon emission+Vacuum polarization diagrams. It is shown that the contribution from the Box+Vertex correction+Soft photon emission+Vacuum polarization diagrams is about 15–18%, and 16–19%, respectively.

We present our results on the dependence of the total cross section as a function of the e^+e^- center-of-mass energy $\sqrt{s}$

Fig. 13 The total cross section of the $e^+ + e^- \rightarrow e^+ + e^- + \mu^+ + \mu^-$ process as a function of $\sqrt{s}$. The experimental data are from [92] of the L3 Collaboration. **a** $6 \text{ GeV} \leq \sqrt{s} \leq 209 \text{ GeV}$; **b** $100 \text{ GeV} \leq \sqrt{s} \leq$

209 GeV - compared to experimental data; **c** $3 \text{ GeV} \leq W_{\gamma\gamma} \leq 40 \text{ GeV}$ and $|\cos\theta_\mu| \leq 0.8$; **d** $150 \text{ GeV} \leq \sqrt{s} \leq 209 \text{ GeV}$, $3 \text{ GeV} \leq W_{\gamma\gamma} \leq 40 \text{ GeV}$ and $|\cos\theta_\mu| \leq 0.8$ - compared to experimental data

in the region $5 \text{ GeV} \leq \sqrt{s} \leq 209 \text{ GeV}$ in Fig. 13a. According to our calculation, we see that the total cross sections in the region of the center-of-mass energy $5 \text{ GeV} \leq \sqrt{s} \leq 150 \text{ GeV}$ increase sharply. The total cross sections increase smoothly and slowly with increasing energy in the region $150 \text{ GeV} \leq \sqrt{s} \leq 209 \text{ GeV}$. According to the experimental data [92], we made an analogous calculation for a specific range of center-of-mass energies $150 \text{ GeV} \leq \sqrt{s} \leq 209 \text{ GeV}$, in order to compare with these experimental data. In Fig. 13b, we show our results. It is seen from Fig. 13b that our theoretical results are in satisfactory agreement with experimental data.

In Fig. 13c, d, we present our results for these cases. We plot the dependence of the cross section of the $e^+ + e^- \rightarrow e^+ + e^- + \mu^+ + \mu^-$ (35) process as a function of the e^+e^- centre-of-mass energy in the region $5 \text{ GeV} \leq \sqrt{s} \leq 209 \text{ GeV}$ at the $3 \text{ GeV} \leq W_{\gamma\gamma} \leq 40 \text{ GeV}$ and $(-0.8 \leq \cos\theta_\mu \leq 0.8)$ in Fig. 13c. Our calculations show that the total cross sections in the region of the e^+e^- center-of-mass energy $5 \text{ GeV} \leq \sqrt{s} \leq 110 \text{ GeV}$ increase sharply. The total cross sections increase

smoothly and slowly with increasing energy in the region $110 \text{ GeV} \leq \sqrt{s} \leq 209 \text{ GeV}$. To compare with experimental data [92], we perform analogous calculations for a specific range of the center-of-mass energy $150 \text{ GeV} \leq \sqrt{s} \leq 209 \text{ GeV}$. Figure 13d shows that our theoretical results are in satisfactory agreement with the experimental data.

It should be noted that during the calculation for order of $O(\alpha_{em}^4)$, we see that the contribution of the Box diagram to the $e^+ + e^- \rightarrow e^+ + e^- + \mu^+ + \mu^-$ process is very small, that is about 2–3%.

In fact, all the contributions from “the Box+Vertex correction+Soft photon emission+ Vacuum polarization diagrams” that we are considering are also taken into account for the $e^+ + e^- \rightarrow e^+e^-\mu^+\mu^-$ process. For the process $e^+ + e^- \rightarrow e^+e^-\mu^+\mu^-$ in L3 LEP [92] studied for the Born approximation. But, it should be noted that the contributions of all these extra corrections for the $e^+ + e^- \rightarrow e^+e^-\mu^+\mu^-$ process are very small, that is about 2–3%. Maybe this one is dependent from energy.

If the future colliders for electron–positron collisions will have higher energies, then there may be contributions from these corrections and even contributions from other corrections will be effective.

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