ELEMENTARY PARTICLES AND FIELDS

Theory

Landau–Khalatnikov–Fradkin Transformation and Even ζ Functions

A. V. Kotikov^{1)*} and S. Teber²⁾

Received April 13, 2020; revised April 13, 2020; accepted April 13, 2020

Abstract—An exact formula that relates standard ζ functions and so-called hatted ζ ($\hat{\zeta}$) functions in all orders of perturbation theory is presented. This formula is based on the Landau–Khalatnikov–Fradkin transformation.

DOI: 10.1134/S1063778820060198

1. INTRODUCTION

We consider properties of multiloop massless functions of the propagator type. There is an ever growing number of indications (see, for example, [1]) that, in the calculations of various quantities in the Euclidean region, striking regularities arise in terms proportional to ζ_{2n} —that is, to even Euler ζ functions. These regularities are thought [2] to be due to the fact that ε -dependent combinations of ζ functions, such as

$$\begin{aligned}\hat{\zeta}_3 &\equiv \zeta_3 + \frac{3\varepsilon}{2}\zeta_4 - \frac{5\varepsilon^3}{2}\zeta_6, \\ \hat{\zeta}_5 &\equiv \zeta_5 + \frac{5\varepsilon}{2}\zeta_6, \quad \hat{\zeta}_7 \equiv \zeta_7,\end{aligned}\tag{1}$$

rather than the ζ functions themselves are dominant objects that eliminate ζ_{2n} in the ε expansions of four-loop functions belonging to the propagator type. A generalization of combinations in (1) to the cases of five, six, and seven loops can be found in [3].³⁾ The results in (1) and their generalization in [3] make it possible to predict π^{2n} terms in higher orders of perturbation theory.

In [4] (see also [5]), the present authors extended the results in (1) to any order in ε in a rather unexpected way—by means of the Landau–Khalatnikov–Fradkin (LKF) transformation [6], which relates the fermion propagators in quantum electrodynamics (QED) in two different gauges. It should be noted

that the most important applications of the LKF transformation are generally associated with the predictions of some terms in high orders of perturbation theory in QED [7], its generalizations [8], and more general $SU(N)$ gauge theories [9].

In the present article, we give a brief survey of the results reported in [4], placing emphasis on how the LKF transformation demonstrates in a natural way the existence of $\hat{\zeta}$ functions and makes it possible to extend the results in (1) to any order in ε .

2. LKF TRANSFORMATION

Let us consider QED in d -dimensional ($d = 4 - 2\varepsilon$) Euclidean space. In general, the fermion propagator in a gauge involving the parameter ξ in the p and x representations has the form

$$S_F(p, \xi) = \frac{1}{i\hat{p}}P(p, \xi), \quad S_F(x, \xi) = \hat{x}X(x, \xi), \tag{2}$$

where there are explicit expressions for the factors $\hat{p}$ and $\hat{x}$, which involve the Dirac γ matrices.

Within the dimensional regularization, the LKF transformation relates the fermion propagator in these two gauges with parameters ξ and η , respectively, as [4]

$$S_F(x, \xi) = S_F(x, \eta)e^{iD(x)}, \tag{3}$$

where

$$D(x) = \frac{i\Delta A}{\varepsilon}\Gamma(1 - \varepsilon)(\pi\mu^2 x^2)^\varepsilon, \tag{4}$$

$$\Delta = \xi - \eta, \quad A = \frac{\alpha_{\text{em}}}{4\pi} = \frac{e^2}{(4\pi)^2}.$$

This means that $D(x)$ makes a contribution proportional to ΔA and the pole ε^{-1} .

¹⁾Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna, Russia.

²⁾Sorbonne Université, CNRS, Laboratoire de Physique Théorique et Hautes Energies, LPHE, F-75005 Paris, France.

*E-mail: kotikov@theor.jinr.ru

³⁾We note that the results in [3] also contain multiple ζ functions (multi-zeta values), but their analysis is beyond the scope of the present article.

Suppose that, for a gauge-fixing parameter η , the fermion propagator $S_F(p, \eta)$ with an external momentum p has the form (2), where

$$P(p, \eta) = \sum_{m=0}^{\infty} a_m(\eta) A^m \left(\frac{\tilde{\mu}^2}{p^2} \right)^{m\varepsilon}, \quad (5)$$

$$\tilde{\mu}^2 = 4\pi\mu^2.$$

Here, $a_m(\eta)$ are the coefficients in the loop expansion of the propagator and $\tilde{\mu}$ is the renormalization scale lying between the scales of the MS (minimal-subtraction) and $\overline{\text{MS}}$ (modified-minimal-subtraction) schemes. The LKF transformation determines the fermion propagator for another gauge parameter ξ as

$$P(p, \xi) = \sum_{m=0}^{\infty} a_m(\eta) A^m \left(\frac{\tilde{\mu}^2}{p^2} \right)^{m\varepsilon} \times \sum_{l=0}^{\infty} \frac{1 - (m+1)\varepsilon}{1 - (m+l+1)\varepsilon} \times \Phi_{\text{MV}}(m, l, \varepsilon) \frac{(\Delta A)^l}{(-\varepsilon)^l l!} \left(\frac{\mu_{\text{MV}}^2}{p^2} \right)^{l\varepsilon}, \quad (6)$$

where

$$\Phi_{\text{MV}}(m, l, \varepsilon) = \frac{\Gamma(1 - (m+1)\varepsilon) \Gamma(1 + (m+l)\varepsilon) \Gamma^{2l}(1 - \varepsilon)}{\Gamma(1 + m\varepsilon) \Gamma(1 - (m+l+1)\varepsilon)}. \quad (7)$$

Here, the symbol MV stands for the so-called minimal Vladimirov scale introduced in [4]. We note that, in [4], the use of the popular G scale [10] led to the same final results given in Eqs. (16) and (17) below.

In order to derive expression (6), we employed the fermion propagator $S_F(p, \eta)$ with $P(p, \eta)$ given by (5), applied the Fourier transformation to $S_F(x, \eta)$, and made the LKF transformation (3). As a final step, we performed the inverse Fourier transformation and obtained the fermion propagator $S_F(p, \xi)$ with $P(p, \xi)$ given in (6).

Let us now study the factor $\Phi_{\text{MV}}(m, l, \varepsilon)$. For this, we make use of the expansion of the Γ function in the form

$$\Gamma(1 + \beta\varepsilon) = \exp \left[-\gamma\beta\varepsilon + \sum_{s=2}^{\infty} (-1)^s \eta_s \beta^s \varepsilon^s \right], \quad (8)$$

$$\eta_s = \frac{\zeta_s}{s},$$

where γ is the Euler constant. Substituting this expansion into expression (7), we recast the factor $\Phi_{\text{MV}}(m, l, \varepsilon)$ into the form

$$\Phi_{\text{MV}}(m, l, \varepsilon) = \exp \left[\sum_{s=2}^{\infty} \eta_s p_s(m, l) \varepsilon^s \right], \quad (9)$$

where

$$p_s(m, l) = (m+1)^s - (m+l+1)^s + 2l + (-1)^s \left\{ (m+l)^s - m^s \right\}, \quad (10)$$

$$p_1(m, l) = 0, \quad p_2(m, l) = 0.$$

One can readily see from Eq. (9) that the factor $\Phi_{\text{MV}}(m, l, \varepsilon)$ involves values of the ζ_s function of given weight s (or transcendental level) in front of ε^s . This property constrains strongly the coefficients, thereby simplifying the ensuing analysis (the authors of the articles quoted in [11] also used this property).

3. $\hat{\zeta}_{2n-1}$

We now focus on the polynomial $p_s(m, l)$ in Eq. (10). It is convenient to partition it into components featuring even and odd values of s . The following recursion relations hold:

$$p_{2k} = p_{2k-1} + L p_{2k-2} + p_3, \quad (11)$$

$$p_{2k-1} = p_{2k-2} + L p_{2k-3} + p_3, \quad L = l(l+1).$$

Expressing even components, p_{2k} , in terms of odd ones as

$$p_{2k} = \sum_{s=2}^k p_{2s-1} C_{2k, 2s-1} \quad (12)$$

$$= \sum_{m=1}^{k-1} p_{2k-2m+1} C_{2k, 2k-2m+1},$$

we can determine the exact structure of $C_{2k, 2k-2m+1}$ in the form

$$C_{2k, 2k-2m+1} = b_{2m-1} \frac{(2k)!}{(2m-1)!(2k-2m+1)!}, \quad (13)$$

$$b_{2m-1} = \frac{(2^{2m} - 1)}{m} B_{2m},$$

where B_m are well-known Bernoulli numbers.

It is now convenient to represent the argument of the exponential form on the right-hand side of Eq. (9) in the form

$$\sum_{s=3}^{\infty} \eta_s p_s \varepsilon^s = \sum_{k=2}^{\infty} \eta_{2k} p_{2k} \varepsilon^{2k} + \sum_{k=2}^{\infty} \eta_{2k-1} p_{2k-1} \varepsilon^{2k-1}. \quad (14)$$

With the aid of Eq. (12), the first term on the right-hand side of (14) can be represented in the form

$$\sum_{k=2}^{\infty} \eta_{2k} p_{2k} \varepsilon^{2k} \quad (15)$$

$$\begin{aligned}
 &= \sum_{k=2}^{\infty} \eta_{2k} \varepsilon^{2k} \sum_{s=2}^k p_{2s-1} C_{2k,2s-1} \\
 &= \sum_{s=2}^{\infty} p_{2s-1} \sum_{k=s}^{\infty} \eta_{2k} C_{2k,2s-1} \varepsilon^{2k}.
 \end{aligned}$$

Relation (14) can then be recast into the form

$$\begin{aligned}
 &\sum_{s=2}^{\infty} \hat{\eta}_{2s-1} p_{2s-1} \varepsilon^{2s-1} \\
 &= \sum_{s=2}^{\infty} [\hat{\zeta}_{2s-1} / (2s-1)] p_{2s-1} \varepsilon^{2s-1},
 \end{aligned}$$

where

$$\hat{\zeta}_{2s-1} = \zeta_{2s-1} + \sum_{k=s}^{\infty} \zeta_{2k} \hat{C}_{2k,2s-1} \varepsilon^{2(k-s)+1} \quad (16)$$

with

$$\begin{aligned}
 \hat{C}_{2k,2s-1} &= \frac{2s-1}{2k} C_{2k,2s-1} \\
 &= b_{2k-2s+1} \frac{(2k-1)!}{(2s-2)!(2k-2s+1)!}.
 \end{aligned} \quad (17)$$

Relations (16), (17), and (13) lead to an expression for $\hat{\zeta}_{2s-1}$ in terms of standard ζ functions that is valid in all orders of the expansion in ε .

4. CONCLUSIONS

The recursion relations in (11) between the even and odd components of the polynomial associated with the factor $\Phi_{MV}(m, l, \varepsilon)$ (7) have been deduced from the result in (6) obtained by means of the LKF transformation for the fermion propagator. These recursion relations make it possible to express all results for the factor $\Phi_{MV}(m, l, \varepsilon)$ in terms of $\hat{\zeta}_{2s-1}$. Expressions (16) and (17) for them are valid in any order of perturbation theory.

ACKNOWLEDGMENTS

A.V. Kotikov is grateful to the Organizing Committee of the Session-Conference of Nuclear Physics Section at the Department of Physical Sciences, Russian Academy of Sciences, for the invitation.

REFERENCES

1. P. A. Baikov and K. G. Chetyrkin, PoS (LL2018), 008 (2018).
2. D. J. Broadhurst, hep-th/9909185; P. A. Baikov and K. G. Chetyrkin, Nucl. Phys. B **837**, 186 (2010).
3. A. Georgoudis, V. Goncalves, E. Panzer, and R. Pereira, arXiv: 1802.00803 [hep-th]; P. A. Baikov and K. G. Chetyrkin, J. High Energy Phys. **1806**, 141 (2018); **1910**, 190 (2019).
4. A. V. Kotikov and S. Teber, Phys. Rev. D **100**, 105017 (2019).
5. A. V. Kotikov and S. Teber, arXiv: 1912.10957 [hep-th].
6. L. D. Landau and I. M. Khalatnikov, Sov. Phys. JETP **2**, 69 (1956); E. S. Fradkin, Sov. Phys. JETP **2**, 361 (1956).
7. A. Bashir and A. Raya, Phys. Rev. D **66**, 105005 (2002); S. Jia and M. R. Pennington, Phys. Rev. D **95**, 076007 (2017).
8. A. Ahmad, J. J. Cobos-Martínez, Y. Concha-Sánchez, and A. Raya, Phys. Rev. D **93**, 094035 (2016); A. James, A. V. Kotikov, and S. Teber, Phys. Rev. D **101**, 045011 (2020).
9. T. De Meerleer, D. Dudal, S. P. Sorella, P. Dall'Olivo, and A. Bashir, Phys. Rev. D **97**, 074017 (2018); **101**, 085005 (2020).
10. K. G. Chetyrkin, A. L. Kataev, and F. V. Tkachov, Nucl. Phys. B **174**, 345 (1980).
11. A. V. Kotikov and L. N. Lipatov, Nucl. Phys. B **582**, 19 (2000); **661**, 19 (2003); B **769**, 217 (2007); A. V. Kotikov, L. N. Lipatov, A. I. Onishchenko, and V. N. Velizhanin, Phys. Lett. B **595**, 521 (2004); L. Bianchi, V. Forini, and A. V. Kotikov, Phys. Lett. B **725**, 394 (2013).