Concept of Coherence in Neutrino and Antineutrino Scattering off Nuclei

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Abstract—The concept of coherence in the scattering of neutrinos and antineutrinos off nuclei is discussed. Motivated by the results of the COHERENT experiment, a new approach to coherence in these processes is proposed, which allows a unified description of the elastic (coherent) and inelastic (incoherent) contributions to the total cross section for neutrino and antineutrino scattering off nuclei at energies below 100 MeV. Experiments and physical problems for coherent scattering of (anti)neutrinos off nuclei are briefly discussed. The extended appendix covers the main points and conclusions of the proposed approach in pedagogical detail.

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1. INTRODUCTION: IS THERE COHERENCE IN THE COHERENT EXPERIMENT?

The so-called coherent neutrino scattering off complex composite systems (nuclei and atoms) has long been considered an important source of information about neutrino physics, including that beyond the current Standard Model. The main attractive feature of this phenomenon is a quite noticeable (a few orders of magnitude) increase in its probability as compared to the case of a simple point target. Of particular interest is the fact that neutrinos can coherently scatter not only from nucleons inside nuclei [1–3] but also from whole atoms, including orbital electrons [4–6].

The decisive factor is the quantum-dynamical principle, which consists in that the individual amplitudes of the neutrino–nucleon scattering are summed with the identical phase (i.e., coherently), which results in an appreciable increase in the probability of this process roughly proportional to the squared number of neutrons in the nucleus. The commonly accepted quantum-mechanical condition of coherence is the inequality

$$qR \ll 1, \quad (1)$$

where q is the three-momentum transferred from the neutrino to the nucleus whose characteristic size is determined by radius R . Roughly, the wavelength of the incident neutrino $1/q$ is noticeably larger than the size R of the obstacle in its way; therefore, the neutrino is unable to distinguish structural details of the obstacle and “feels the nucleus as a whole.”

Since the characteristic size of nuclei is not much beyond the range of $1/25$ to $1/150$ MeV, the above coherence condition is well fulfilled for a large number of available sources of neutrinos (antineutrinos), such as reactor antineutrinos, solar neutrinos, (anti)neutrinos from supernova explosions, and accelerator neutrinos [7]. This gives particular significance to coherent neutrino scattering off nuclei. The range of physical problems that can be solved through careful study of coherent neutrino and antineutrino scattering from nuclei is discussed in more detail in Section 8.

Nevertheless, a neutrino with an energy of about 40 MeV gives rise to an insignificant nuclear recoil energy (in the keV region), which is the only observable quantity of coherent scattering that has not yet been measured directly. However, in 2017 the COHERENT experiment “bypassed the problem” by using a high-energy (anti)neutrino flux.

Fig. 1. Observation of Coherent Elastic Neutrino–Nucleus Scattering (CENS) according to the COHERENT experiment [8]. Experimental points (right) are the difference of the CsI[Na] detector counts with the neutrino beam on and off as a function of energy (number of photoelectrons). The figure is borrowed from [8].

Recall that the main goal of the COHERENT collaboration was to detect and study Coherent Elastic Neutrino–Nucleus Scattering (CEvNS). This was supposed to do with a powerful pulsed neutrino beam with an energy in the range of 10–55 MeV produced on the basis of the Spallation Neutron Source (SNS) at the Oak Ridge National Laboratory. In 2017, the COHERENT collaboration [8] claimed the first observation of CEvNS (Fig. 1) using the 14.6-kg CsI[Na] scintillation detector [9]. The results and plans of the COHERENT collaboration are presented in more detail in [8, 10–18]. Discussing the formulation of the problem, the authors of the COHERENT experiment [8] were fully aware of that “...coherent interactions between neutrinos and all nucleons present in an atomic nucleus would exist only as long as the momentum exchanged remained *significantly smaller* than the inverse of the nuclear size...” However, simple estimations show that the COHERENT data underlying the coherent scattering detection claim fail to meet condition (1).

Indeed, kinetic energy of the scattered nucleus is given by the expression $T_A \approx \frac{q^2}{2M_A}$, where M_A is the mass of the nucleus A . Considering that the mass of the cesium-133 nucleus in natural units is $M = 123.7$ GeV, for the transferred momentum we have

$$q = 15.7 \text{ MeV} \left(\frac{T_A}{\text{keV}} \right)^{1/2} = 14.5 \text{ MeV} N_{\text{p.e.}}^{1/2}, \quad (2)$$

where the second equality allows for the fact that the recoil energy of the cesium nucleus $T_A = 1$ keV corresponds to approximately 1.17 photoelectrons (p.e.) [8]. Since the radius of the nucleus with the atomic number A can be expressed as $R_A \approx 1.2 \times A^{1/3}$, for ^{133}Cs there is

$$R \approx 6.13 \text{ fm} = 0.031 \text{ MeV}^{-1}. \quad (3)$$

Therefore, from (2) and (3) one has

$$qR(N_{\text{p.e.}}) = 0.45N_{\text{p.e.}}^{1/2}. \quad (4)$$

In Fig. 1 borrowed from [8], the measured spectrum that served as a basis for detection of coherent neutrino–nucleus scattering is shown as a function of detected photoelectrons. It begins with $N_{\text{p.e.}} = 5$, reaches the maximum at $N_{\text{p.e.}} \approx 14$, and completely disappears at $N_{\text{p.e.}} \approx 35$. Hence, according to (4), it follows that

$$\begin{aligned} qR(N_{\text{p.e.}} = 5) &= 1, \quad qR(N_{\text{p.e.}} = 14) = 1.69, \\ qR(N_{\text{p.e.}} = 35) &= 2.66, \end{aligned} \quad (5)$$

and condition (1), is seen not to be fulfilled for all COHERENT data. What is the result of this naïve estimation?

Though, according to (5), condition (1) is obviously not fulfilled in the COHERENT experiment [8] and the transferred momentum is NOT (considerably) smaller than the inverse of the size of the nucleus, it cannot be asserted that the collaboration could not observe elastic coherent neutrino–nucleus scattering. What did they actually “observe”?

The answer is that the collaboration observed events of neutral-current neutrino–nucleus interaction in the MeV energy region, which itself is an important achievement. However, the data cannot be unambiguously considered as pure CEvNS, which the authors believe them to be [8]. These data show a noticeable (10–20%) contribution from inelastic (anti)neutrino–nucleus interactions utterly indistinguishable from CEvNS, since only the nuclear recoil energy is detected [19, 20].

If there were not the COHERENT experiment, we would probably never know how coherent neutrino scattering off a complex system smoothly and naturally passes into incoherent one. The search for a quantitative answer to this question stimulated the work that resulted in this review.

Fig. 2. “Appearance” of coherent (elastic) νA scattering by the exchange of the neutral Z^0 boson (left). Individual (independent) scattering centers are located at fixed points $\mathbf{x}_j$ in the composite target A (right). It is this spatial “separateness” that, according to [3], leads to occurrence of the phase factor $e^{i\mathbf{q} \cdot (\mathbf{x}_j - \mathbf{x}_i)}$, which is decisive for the presence or absence coherence.

2. EVOLUTION OF THE CONCEPT OF COHERENCE IN NEUTRINO SCATTERING

2.1. Coherent Elastic Neutrino–Nucleus Scattering According to Freedman

It is believed that Freedman was the first to discuss coherent neutrino scattering off a system composed of nucleons (nucleus) [1, 3]. In their little known work [2], Kopeliovich and Frankfurt also discussed the coherent effect in neutrino–nucleus scattering.

We briefly recall Freedman’s logic for reasoning about the coherent effect in scattering of neutrinos off nuclei [1, 3]. In [3], it was pointed out that a noticeable increase in the cross section for the process $\nu + A \rightarrow \nu + A$ (Fig. 2, left) is a consequence of the main quantum mechanics principle in the sense that this phenomenon always occurs when a structureless incident particle (neutrino) elastically scatters off a composite system (nucleus) which is supposed to consist of A mutually independent scattering centers (nucleons), each of which is located at a fully fixed space point $\mathbf{x}_i$, where $i = 1, \dots, A$ (Fig. 2, right).

In compliance with the quantum-mechanical principle of superposition, the overall amplitude of the neutrino scattering off a composite object $M(\mathbf{k}', \mathbf{k})$, where $\mathbf{k}$ and $\mathbf{k}'$ are the initial and final neutrino momenta (Fig. 2, left), is a sum of individual amplitudes of neutrino scattering off independent scatterers in the composite object $f_j(\mathbf{k}', \mathbf{k})$ taken with the phase factor $e^{i(\mathbf{k}' - \mathbf{k}) \cdot \mathbf{x}_j}$ that allows for the (relative) variation in the phase of the incident (neutrino) wave in the scat-

tering off individual scatterers located at different space points $\mathbf{x}_j$ (Fig. 2, right)

$$\begin{aligned} M(\mathbf{k}', \mathbf{k}) &= \sum_{j=1}^A f_j(\mathbf{k}', \mathbf{k}) \exp i(\mathbf{k}' - \mathbf{k}) \cdot \mathbf{x}_j \\ &= \sum_{j=1}^A f_j(\mathbf{k}', \mathbf{k}) e^{i\mathbf{q} \cdot \mathbf{x}_j}. \end{aligned} \quad (6)$$

Here $\mathbf{q} = \mathbf{k}' - \mathbf{k}$ is the neutrino momentum transferred to the composite object or lost [3]. Due to smallness of individual ν scattering amplitudes $f_j(\mathbf{k}', \mathbf{k})$ (weak interaction), terms responsible for possible (successive) scattering off several scatterers like $\propto f_i(\mathbf{k}', \mathbf{k}) f_j(\mathbf{k}', \mathbf{k})$ are omitted in (6). Since there are phase factors $e^{i\mathbf{q} \cdot (\mathbf{x}_j - \mathbf{x}_i)}$, the differential cross section

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= |M(\mathbf{k}', \mathbf{k})|^2 = \sum_{j=1}^A |f_j(\mathbf{k}', \mathbf{k})|^2 \\ &+ \sum_{j \neq i}^A \sum_{i=1}^A f_i(\mathbf{k}', \mathbf{k}) f_j^*(\mathbf{k}', \mathbf{k}) e^{i\mathbf{q} \cdot (\mathbf{x}_j - \mathbf{x}_i)}, \end{aligned} \quad (7)$$

appreciably depends on the relation between the value of the transferred momentum $\mathbf{q}$ ($q = |\mathbf{q}| = 2|\mathbf{k}| \sin \frac{\theta}{2}$, where θ is the neutrino scattering angle) and the size of the system of scatterers $R = \max_{ij} |\mathbf{x}_i - \mathbf{x}_j|$.

According to [3], if the dimensionless quantity $qR \approx 1$ (or $qR \geq 1$), all independent scattering centers (nucleons) are too far apart in space, and relative phase factors are important (because of being different). As a result, there occurs considerable cancellation of terms in the second (nondiagonal) $A(A-1)$ term, and it makes almost no contribution to cross section (7), which turns out to be small due to small-

ness of the amplitudes $f_j(\mathbf{k}', \mathbf{k})$ and is (only) proportional to A .

On the other hand [3], if $qR \ll 1$ (i.e., q is considerably smaller than the inverse of the target size), all phase factors $e^{iq \cdot (\mathbf{x}_j - \mathbf{x}_i)} \approx 1$ (i.e., relative phases are unimportant because they are almost equal to one another), and contributions from all independent scatterers are added coherently in the second sum of (7). As a result, the total cross section (called coherent in [3]) turns out to be amplified by a factor of A^2

$$\frac{d\sigma}{d\Omega} \approx A^2 |\bar{f}(\mathbf{k}', \mathbf{k})|^2, \quad (8)$$

where $\bar{f}(\mathbf{k}', \mathbf{k}) \equiv \frac{1}{A} \sum_j f_j(\mathbf{k}', \mathbf{k})$.

If all scatterers are identical, then $\bar{f}(\mathbf{k}', \mathbf{k}) = f_j(\mathbf{k}', \mathbf{k})$, and the coherent cross section is really A^2 times larger than the cross section of the scattering off an individual scatterer ($\propto |\bar{f}(\mathbf{k}', \mathbf{k})|^2$). This is the main point of the concept of coherence in neutrino–nucleus scattering according to Freedman [3].

Note that the answer to the question of how to correctly calculate the cross section for the neutrino–nucleus scattering—by taking only the first term in (7) or by using (8)—depends on establishing how the dimensionless quantity qR is related to unity ($qR \ll 1$ or $qR \gg 1$); i.e., it depends on which of the two limiting cases is realized. This is qualitatively clarified prior to the calculation of the cross section. If $qR \gg 1$, there is one procedure; if vice versa, the other. In other words, beginning with [3], in all subsequent publications the first to be done was almost always to qualitatively determine (assume, declare) that the coherence condition is fulfilled (i.e., $qR \ll 1$), and a formula like (8) can be employed.

Then the author [3] writes the following: “... we assumed in the previous discussion that individual constituents are located at definite points $\mathbf{X}_j$ in the composite target. This restriction is inessential and the discussion can be generalized to the case where the target is treated as a quantum-mechanical system. There is one point of principle concerning the various quantum states, such as charge and spin projection, of the individual constituents of the composite system: namely, the instruction to sum over constituent amplitudes in Equation 1.1 follows from the superposition principle only if there is no change in quantum states. Equation 1.1 does not apply to spin-flip and charge-exchange processes, and these cross sections are not amplified by a factor of A^2 in a composite target.” However, all these statements are made at a qualitative level, that is, without proof. Moreover, in some current works, as discussed below in Section 8, they are simply ignored.

To complete the picture, we continue setting forth Freedman’s logic in [3]. If we assume that all scatterers (constituents) are identical ($f_j(\mathbf{k}', \mathbf{k}) = f(\mathbf{k}', \mathbf{k})$) and pointlike, and that the spatial density for them has the form $\rho(\mathbf{x}) = \sum_j \delta(\mathbf{x} - \mathbf{x}_j)$, i.e., they are really “nailed” at definite points in the composite target, then we can remove summation over j in (6) and obtain the expression

$$M(\mathbf{k}', \mathbf{k}) = f(\mathbf{k}', \mathbf{k}) \int d^3\mathbf{x} \exp(i\mathbf{q} \cdot \mathbf{x}) \rho(\mathbf{x}) = f(\mathbf{k}', \mathbf{k}) F(\mathbf{q}). \quad (9)$$

Therefore, it can be stated that coherent scattering “measures” the composite target form factor

$$F(\mathbf{q}) \equiv \rho(\mathbf{q}) = \int d^3\mathbf{x} e^{i\mathbf{q} \cdot \mathbf{x}} \rho(\mathbf{x}), \quad (10)$$

which is a Fourier transform of the (material) density distribution in this target. For the spherically symmetric nuclear Gaussian density with $R = r_0 A^{1/3}$ and $r_0 \approx 10^{-13}$ cm

$$\rho(r) = A(4\pi R^2)^{-3/2} \exp\left(-\frac{r^2}{4R^2}\right)$$

one has $F(q) = \rho(q) = A \exp(-q^2 R^2)$.

As a result, the behavior of the cross section can be described at the qualitative level as follows. Since for the beam momentum k the maximum transferred momentum is $q_{\max} = 2k$, the cross section $d\sigma/d\Omega$ (8) at the intermediate or higher energies ($2kR \leq 1$) has a pronounced forward peak and rapidly decreases, due to the form factor $F(\mathbf{q})$, with increasing emission angle θ (or q). Under these conditions the total elastic cross section

$$\sigma_{\text{el}} = \int d\Omega \frac{d\sigma}{d\Omega} \sim A^{4/3} \quad (11)$$

has an “unimportant” $A^{4/3}$ behavior. However, at low energies the coherence condition $2kR \ll 1$ is satisfied for all scattering angles, and due to $F(\mathbf{q}) \approx \text{constant}$, the angular dependence of the cross section is entirely determined by the individual amplitude $f(\mathbf{k}', \mathbf{k})$ in (9). The coherence effect is fully manifested at low energies, and $\sigma_{\text{el}} \propto A^2$. This conclusion is largely independent of the nature of the incident particle [3].

Rejection of some simplifying assumptions used in the above discussion allows the concept of coherent scattering to be used more extensively. For example, in crystals and complex molecules the cross section peak (6) manifests itself at nonzero transferred momenta, which is a consequence of regular arrangement of individual scatterers. An example is the Bragg peak in the scattering of X rays from biological objects [3]. Electron–nucleus scattering [21] is a well-studied usage of the coherent scattering formalism.

If individual scatterers are different, as in the case of protons and neutrons with different projections of spin in the nucleus, noticeable cancellation of contri-

butions is possible in (8). Without thorough knowledge of characteristics of individual amplitudes, it is impossible to predict the magnitude of the coherent cross section. However, with extreme cancellations ignored, it makes sense to expect that the coherent scattering cross section is approximately a factor A^2 amplified as compared to the cross section for the scattering off a separate scattering center [3].¹

The main ideas of [3] were further developed. For example, in [22] coherent differential cross section (7) for spin-zero nuclei ($J = 0$) is already given in the modern form

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\phi d(\cos\theta)} = \frac{G_F^2 E_\nu^2}{4(2\pi)^2} \times (1 + \cos\theta) \left[(1 - 4\sin^2\theta_W)Z - N \right]^2. \quad (13)$$

The elastic scattering angle θ is related to the nuclear recoil energy T , and therefore the distribution over the kinetic nuclear recoil energy has the form

$$\frac{d\sigma}{dy} = \frac{G_F^2}{2\pi} \frac{M(M + 2E_\nu)^2}{[M + 2E_\nu(1 - y)]^3} \times E_\nu^2(1 - y) \left[(1 - 4\sin^2\theta_W)Z - N \right]^2, \quad (14)$$

¹ In [3], the role of coherent neutrino scattering off the nuclear medium in supernova evolution is discussed. Formulas are given for the overall amplitude of coherent scattering off spin-zero ($J = 0$) nuclei with $Z = N$

$$M(\mathbf{k}', \mathbf{k}) \sim G_A e^{-q^2 R^2} (-\sin^2\theta_W) \bar{u}_l(v') \gamma^0 u_\nu(v), \quad (12)$$

and for differential and total cross sections

$$\frac{d\sigma_A}{d\cos\theta} \approx \frac{\sigma_0}{8} (-\sin^2\theta_W)^2 A^2 \omega^2 (1 + \cos\theta),$$

$$\sigma_A \approx \frac{\sigma_0}{4} (-\sin^2\theta_W)^2 A^2 \omega^2.$$

The neutrino energy is assumed to be $\omega \ll m_A$ (point nucleus). If $J \neq 0$ and $Z \neq N$, it is necessary to sum over individual nucleons

$$M(\mathbf{k}', \mathbf{k}) \sim G e^{-q^2 R^2} \left[-\sin^2\theta_W (Z + N) + \frac{1 - 2\sin^2\theta_W}{2} (Z - N) - \frac{g_V}{2} (Z_\uparrow - Z_\downarrow) + \frac{g_A}{2} (N_\uparrow - N_\downarrow) \right] \bar{u}(v') \gamma^0 u(v),$$

where the arrow indicates nucleons with spins directed upward and downward that are scattered via axial neutral current. For all nuclei (except very light ones) the most important term in $M(\mathbf{k}', \mathbf{k})$ is proportional to $(-\sin^2\theta_W)(Z + N)$, in other words, relation (13) is usually valid. For spin-zero ($J = 0$) nuclei with $Z \neq N$ there is a relation to the nuclear isospin $I_3 = \frac{1}{2}(Z - N)$ in the form

$$\frac{d\sigma_A^{J=0}}{d\cos\theta} \approx \frac{\sigma_0}{8} A^2 \sin^4\theta_W \left[1 + \frac{1 - 2\sin^2\theta_W}{-\sin^2\theta_W} \frac{I_3}{A} \right]^2 \omega^2 (1 + \cos\theta),$$

$$\sigma_A^{J=0} \approx \frac{\sigma_0}{4} \sin^4\theta_W A^2 [1 + \dots]^2 \omega^2.$$

where G_F is the Fermi constant, E_ν is the neutrino energy, T is the nuclear recoil energy, $y = T/T_{\max} \in [0, 1]$, $T_{\max} = 2E_\nu^2/(M + 2E_\nu)$, and M is the nuclear mass. For $2E_\nu \ll M$, $T_{\max} \approx 2E^2/M$, and the expression [22]

$$\frac{d\sigma}{dy} \approx \frac{G_F^2}{2\pi} E_\nu^2 (1 - y) \left[(1 - 4\sin^2\theta_W)Z - N \right]^2 \approx \frac{G_F^2}{2\pi} E_\nu^2 (1 - y) N^2 \quad (15)$$

is considered to be a very good approximation of formula (14). The right-hand side of (15) is the best known formula of the coherent neutrino–nucleus scattering when (as a rule) the proton contribution is suppressed by a small quantity $(1 - 4\sin^2\theta_W)$ and “coherent” amplification is ensured by the square of the number of neutrons in the nucleus. Moreover, expression (15) assumes that coherence condition (1) is 100% satisfied and the nucleus has no effect at all.

However, this “extremely pure” case is hardly achievable, and the ideal form (15) for the coherent neutrino–nucleus scattering should be modified by considering dependence on the nuclear structure, that is, by including the nuclear form factor, as was the case in the electron–nucleus scattering (see, for example, [21]). On the other hand, inclusion of form factors explicitly or implicitly means deviation from pure coherence.

As an example of considering form factors on the basis of the Freedman principle of superposition of amplitude summation [3], a neutrino–nucleus scattering cross section for $J = 0$ nuclei was obtained in [7] in the form

$$\frac{d\sigma}{dT} = \frac{G_F^2}{2\pi} M \left[2 - \frac{2T}{E_\nu} + \left(\frac{T}{E_\nu} \right)^2 - \frac{MT}{E_\nu^2} \right] \frac{Q_W^2}{4} F^2(q^2), \quad (16)$$

where $Q_W = N - Z(1 - 4\sin^2\theta_W)$ is the weak nuclear charge, $q^2 = 2E_\nu^2 TM/(E_\nu^2 - E_\nu T)$ is the transferred momentum, and $F(q^2)$ is the nuclear form factor. It was proposed, e.g., in [23], to measure the neutron nuclear form factor by studying elastic neutrino–nucleus scattering using an expression like (16). Indeed, if one has the neutron and nuclear density $\rho_{n,p}(r)$, the form factor in [24] is given in the form

$$F(q^2) = \frac{1}{Q_W} \int [\rho_n(r) - (1 - 4\sin^2\theta_W)\rho_p(r)] \times \frac{\sin(qr)}{qr} r^2 dr. \quad (17)$$

Since $(1 - 4\sin^2\theta_W)$ is small, neutrinos are elastically scattered off the spin-zero nucleus, mainly off neutrons, and measurement of the cross section for

this process gives access to the neutron form factor. With the analytical formulas [25] based on the idea that nucleons in a nucleus with radius R have constant density and the nuclear surface thickness is s , the following expression is obtained for the Fourier transform of this density, a form factor [25]:

$$F(q^2) = \frac{3j_1(qR_0)}{qR_0} e^{-\frac{(qs)^2}{2}}, \quad (18)$$

where $R_0^2 = R^2 - 5s^2$. In [24], the total form factor

$$F(q^2) = \frac{1}{Q_W} [NF_n(q^2) - Z(1 - 4\sin^2 \theta_W)F_p(q^2)] \quad (19)$$

was expressed in terms of the neutron and proton form factors $F_n(q^2)$ and $F_p(q^2)$. Formula (19) was used to estimate the neutron radius R_n and the proton radius R_p in [26].

Let us sum up the discussion of “coherence according to Freedman”. The result of the neutrino–nucleus interaction is the consequence of the neutrino scattering off individual nucleons. Each event of scattering off the k th nucleon is roughly described by the amplitude A^k . If these nucleons are assumed to have fixed coordinates (in the nucleus) $\mathbf{x}_k$, then A^k acquires an additional factor $e^{iq\mathbf{x}_k}$, and the overall amplitude A is $\sum_{k=1}^A A^k e^{iq\mathbf{x}_k}$.

In [21], where, in particular, coherent electron scattering was considered, it was assumed for illustrating the emergence of the factor $e^{iq\mathbf{x}}$ that elastic scattering of high-energy electrons off the distributed charge (nucleus) proceeds like Fraunhofer diffraction in optics (Fig. 3). As is known, two (optical) waves are considered coherent if they have an identical frequency and shape, and also a constant phase difference. This coherence leads to constructive or destructive interference. Thus, amplitudes A^k are coherent if all phase factors $q\mathbf{x}_k$ are almost identical. This coherence condition (1) is fulfilled when $|qR_A| \ll 1$, i.e., when the size (of the nucleus) in which nucleons are assumed to be located is much smaller than the wavelength of the incident particle.

Concerning the analogy to light scattering used in [21], we note that in Fraunhofer diffraction (of the light wave, light flow, beam) the positions of the scattering centers are really strictly fixed; i.e., the diffraction grating is a macroscopic object with geometric characteristics (position and size of slits etc.) well-determined in space. However, for the nucleus as a microscopic quantum complex bound system it is impossible in principle to exactly fix positions of scatterers, and, strictly speaking, this logic is unjustified. The analogy used (based on the wave nature of elementary particles) may only serve to qualitatively

Fig. 3. Phase difference ($\propto$ path-length difference) between the ray running through the point $\mathbf{x}$ and the ray running through the point $\mathbf{x} = 0$ is $\Delta\varphi = \frac{2\pi}{\lambda}(l_1 + l_2)$, where $l_1 = |\mathbf{x}| \cos \phi = (\hat{\mathbf{k}}\mathbf{x})$ and $l_2 = |\mathbf{x}| \cos \theta = -(\hat{\mathbf{k}}'\mathbf{x}) \equiv |\mathbf{x}| \cos(\pi - \theta)$. Therefore, $\Delta\varphi = \frac{2\pi}{\lambda}[\hat{\mathbf{k}}\mathbf{x} - \hat{\mathbf{k}}'\mathbf{x}] \equiv q\mathbf{x}$, where $q = \mathbf{k} - \mathbf{k}'$ is the transferred momentum.

understand causes for the appearance of the exponential phase factor. The wave nature of elementary particles is largely determined by the statistical method of our description of their interaction, which unavoidably and fundamentally leans on the obligatory presence of a flow (of a macroscopically large number) of particles.

2.2. Macroscopic Coherence in Neutrino Scattering

Relation between coherent scattering and macroscopic coherence was examined in [27], where the authors addressed the question of whether coherent neutrino scattering can proceed on macroscopic targets, which would result in even greater increase in the neutrino interaction cross section.

Radiative scattering of neutrinos off atomic electrons (or free electrons in a conductor) accompanied by photon emission was studied. This process can be coherent, when the summed electron recoil momentum, i.e., the difference between the momentum lost by the neutrino and the momentum of the emitted photon, is rather small [27]. These radiative processes look attractive because the energy of the emitted photons can be as large as the momentum transferred to the system of electrons, and thus the problem of detecting extremely small recoil energy can be bypassed.

The requirement of macroscopic coherence strongly restricts the accessible phase volume for the scattered particle and the emitted photon. In [27], the authors showed that when scattering occurs via ordinary weak neutral and charged currents, this requirement leads to strong suppression of elementary (or individual) cross sections, and for this reason the requirement of macroscopic coherence leads to a decrease rather than increase in the total cross section. However, if the νe scattering is via the neutrino magnetic moment or the neutrino electric dipole moment, coherence effects can increase the yield of detected

particles (photons). These macroscopic coherence effects would allow detecting neutrinos with an energy of 0.1–3 keV, which are now beyond the experimental reach. The authors assert that a similar coherent amplification mechanism can also work for relativistic particles from the “dark sector” but not for usually considered nonrelativistic massive dark matter particles.

The problem with detection of coherent neutrino scattering off atoms arises due to the atomic recoil energy being very small and can hardly be detected (see, for example, [4, 28]). Indeed, according to [27], coherence requires that the momentum $|\mathbf{q}|$ transferred to the scatterer be smaller than the inverse of the scatterer radius. Only under this condition it will be impossible to determine from which of the target constituents the neutrino was scattered, and the neutrino waves scattered from different constituents will be in (one) phase with one another, which is a necessary condition for coherent scattering. For the neutrino–atom scattering, this means that $|\mathbf{q}| \lesssim 1/(1-3a) \sim 1$ keV, where $a \approx 0.53$ Å is the Bohr radius of the nucleus. For the atom with the number $A \sim 100$ the recoil energy will be $E_{\text{rec}} \approx \frac{q^2}{2m_A} \sim 10^{-5}$ eV, which is about eight orders of magnitude below the current experimental sensitivity [27].

One of the results in [27] is that macroscopic coherence is possible, and the cross section becomes very large only for neutrinos scattered forward at a very small angle, which corresponds to immeasurably small recoil energies of target nuclei. Nevertheless, it is known that macroscopic coherence is easily achieved in scattering of light, X rays, or neutrons from macroscopic targets, since in this case scattered particles are detected rather than target recoil energy.

In [27], Weber’s proposal of the experiment [29–31] on neutrino scattering from a crystal was shown to be untenable. Absence of recoil of individual atoms (in the crystal), which underlay the proposal, is a necessary but not sufficient condition for macroscopic coherence. In addition, it is required that neutrino waves scattered off different centers be in one phase with one another. Scattering amplitudes of an incident particle scattered from a group of scattering centers should contain the corresponding structural factors that describe relative phases in amplitudes corresponding to different scatterers. For elastic neutrino scattering, this structural factor is given by the known expression

$$F(\mathbf{k} - \mathbf{k}') = \sum_{i=1}^N e^{i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{r}_i},$$

where $\mathbf{k}$ and $\mathbf{k}'$ are the momenta of the initial and final neutrinos, $\mathbf{r}_i$ is the coordinate of the i th scatterer, and N is the total number of scatterers in the target. Intro-

ducing the number density of scatterers $\rho(\mathbf{r}) = \sum_i \delta^3(\mathbf{r} - \mathbf{r}_i)$, one can write this structural factor in the form of a standard form factor $F(\mathbf{q}) = \int d^3r e^{i\mathbf{q} \cdot \mathbf{r}} \rho(\mathbf{r})$, where $\mathbf{q} = \mathbf{k} - \mathbf{k}'$ is the momentum transferred to the target. As was mentioned above, these structural factors are critical for scattering coherence, i.e., the answer to the question of whether or not amplitudes of neutrino scattering from different target particles should be summed coherently [27]. Though the exact form of these factors depends on specific features of the target used in the experiment, the pure coherent and/or incoherent regimes can be investigated in a rather general way. Indeed, the square of the modulus of the transition amplitude involves the factor

$$|F(\mathbf{q})|^2 = \sum_{i,j=1}^N e^{i\mathbf{q} \cdot (\mathbf{r}_i - \mathbf{r}_j)}. \quad (20)$$

If the transferred momentum $\mathbf{q}$ meets the condition $\max_{i,j} \{|\mathbf{q}(\mathbf{r}_i - \mathbf{r}_j)|\} \approx |\mathbf{q}|L \ll 1$, where L is the linear size of the target, all phase factors in the above sum can be replaced with unity, which gives $|F(\mathbf{q})|^2 = N^2$. This means that neutrinos scattered off different target constituents are in one phase with one another. In the opposite limit, when $|\mathbf{q}|L \gg 1$, only diagonal ($i = j$) terms of the sum survive, and it turns out that $|F(\mathbf{q})|^2 \approx N$, i.e., there is ordinary dependence of the total cross section on the number of target particles. This is incoherent scattering [27].

In the case of scattering from crystals, it is possible to obtain the macroscopic coherence effect, when the phase difference $\mathbf{q}(\mathbf{r}_i - \mathbf{r}_j)$ in (24) is a multiple of 2π . This leads to the known Bragg condition for diffraction in a crystal

$$2d \sin \vartheta = n\lambda, \quad (21)$$

where d is the distance between the planes in the crystal, ϑ is the angle between the neutrino momentum and the atomic plane (scattering angle $\theta = 2\vartheta$), $\lambda = 2\pi/|\mathbf{k}|$, and n are integers. As in the case of X ray diffraction in a crystal, the intensity of the scattered neutrino wave in the direction of the Bragg peak is proportional to N^2 . It is noticeably different from zero in narrow cones around the Bragg directions with angles $\Delta\Omega \propto N^{-2/3}$ and is almost zero outside the cones. The intensity of the scattered neutrino wave around the Bragg peak is proportional to $N^{4/3}$ [32]. Since scattered neutrinos are undetectable, it is only the recoil momentum of the whole crystal or the force acting on the whole crystal that can in principle be measured experimentally [27].

For the fixed direction of the neutrino momentum relative to the atomic planes of the crystal and for $n \neq 0$, expression (25) “selects” those neutrino ener-

gies which satisfy the Bragg condition. Since Bragg peaks have finite widths, neutrinos in these finite energy intervals $\Delta\omega$ are actually subjected to the Bragg diffraction, but these intervals are very small and proportional to $1/L \propto N^{-1/3}$. As a result, the total momentum transferred to the crystal will be proportional to $N^{4/3} \times N^{-1/3} = N$, exactly as in the case of an amorphous body [32]. Therefore, macroscopic coherence cannot be obtained in Weber's experiments [27].

Thus, the authors of [27] share Freedman's opinion as to when formulas for coherent neutrino scattering off a complex composite system can be used.

2.3. Approach of Gaponov to Coherent Neutrino–Atom Scattering

Let us recall Gaponov and Tikhonov's works [4, 28] in which elastic scattering of low-energy neutrinos off a composite atomic system was investigated. In the opinion of the authors, it is only in the vector version of the weak interaction that the neutrino is coherently scattered off the atomic system as a whole (electrons, protons, and neutrons). In the axial-vector version of the weak interaction, neutrinos are coherently scattered off two atomic subsystems, shell electrons and nucleons in a nucleus, separately. The authors showed that at neutrino energies lower than or about 10 keV (which is below the threshold of inelastic processes²) there was a region of coherent-optical neutrino phenomena (a new region of neutrino physics called neutrino optics by the authors [4, 28]), where elastic neutrino scattering off the atom as a whole plays the main part.

For further consideration, it seems to be worth knowing that in [4, 28] wave functions of the atomic system in the quantum field approach are used. The overall amplitude [4] of the weak interaction of (anti)neutrinos on a complex atomic system (protons, neutrons, and atomic shell electrons) can be recast in the modern form as follows:

$$M(\mathbf{q}) = \frac{G_F}{\sqrt{2}} J_\alpha(\mathbf{q}) \bar{\nu}(\mathbf{k}') \gamma_\alpha (1 + \gamma_5) \nu(\mathbf{k}). \quad (22)$$

Here $\mathbf{q} = \mathbf{k} - \mathbf{k}'$ is the momentum transferred from the neutrino to the nucleus, $q = |\mathbf{q}| = \frac{2E_\nu}{c} \sin \frac{\theta}{2}$, and θ is the neutrino scattering angle. The total atomic

² Perhaps, these works are the first where attention is explicitly paid to the role of inelastic processes in coherent neutrino scattering. In [19], this role can be thought of as being formulated quantitatively.

(nucleons and electrons) current [4] recast in modern notation has the form

$$\begin{aligned} J_\alpha(\mathbf{q}) &= \int \prod_k^{Z_e+Z_A+N} d\mathbf{r}_k \bar{\Psi}(\mathbf{r}_k) \\ &\times \left[\sum_{i=e,p,n} \sum_{k_i}^{Z_e, Z_A, N} (V_\alpha^{k_i} + A_\alpha^{k_i}) e^{i\mathbf{q}\mathbf{r}_{k_i}} \right] \Psi(\mathbf{r}_k) \\ &= \int \prod_k^{Z_e+Z_A+N} d\mathbf{r}_k \sum_{i=e,p,n} e^{i\mathbf{q}\mathbf{r}_{k_i}} \sum_{k_i}^{Z_e, Z_A, N} \bar{\Psi}(\mathbf{r}_k) H_\alpha^{k_i} \Psi(\mathbf{r}_k), \end{aligned}$$

where $V_\alpha^{k_i} = g_V^i \gamma_\alpha^{k_i}$ ($A_\alpha^{k_i} = g_A^i \gamma_\alpha^{k_i} \gamma_5^{k_i}$) is the vector (axial) operator describing the weak interaction of the neutrino with the k_i th particle of the atom,³ and $H_\alpha^{k_i} \equiv \gamma_\alpha^{k_i} (g_V^i + g_A^i \gamma_5^{k_i})$. The vector $\mathbf{r}_{k_i}$ gives the spatial coordinate of the k_i th particle in the atom (of type $\mathbf{x}_i$ in Fig. 2, right). The wave function of the composite system (with spin-spin electron–nucleus interaction ignored) has the form of two independent subsystems

$$\begin{aligned} \Psi(\mathbf{r}_k) &= \Psi_e(\mathbf{r}_{e_i}) \Psi_A(\mathbf{r}_{A_i}) \\ &= \Psi(\mathbf{r}_{e1}, \dots, \mathbf{r}_{eZ_e}) \Psi(\mathbf{r}_{p1}, \dots, \mathbf{r}_{pZ_A}; \mathbf{r}_{n1}, \dots, \mathbf{r}_{nN}). \end{aligned}$$

Each of them has its own fixed spin, but the system as a whole does not have a definite spin, as the authors state in [4]. In their opinion, this leads to disappearance of coherence in the part related to the axial-vector version of weak interaction. The corresponding cross section for elastic neutrino scattering off a complex atomic system is broken into two terms. One of them describes coherent ν scattering as a whole off the entire system via the vector νe and νN interaction, and the other describes scattering off Z_e electrons and A nucleons separately (incoherent in this sense). Nevertheless, in each of the subsystems scattering is coherent for both the vector case and the axial-vector one, because the states of the subsystems do not change [4].

The elastic scattering cross section with the matrix element (26) is obtained, as the authors [4] write, using the standard scheme.⁴ It depends on the energy and the neutrino scattering angle θ in the lab system [4]

³ In the Standard Model, $g_V^e = \frac{1}{2} + 2\sin^2 \theta_W$, $g_V^p = \frac{1}{2} - 2\sin^2 \theta_W$, $g_V^n = -\frac{1}{2}$, $g_A^e = \frac{1}{2}$, $g_A^p = \frac{g_A}{2}$, $g_A^n = -\frac{g_A}{2}$.

⁴ That is, by averaging over the initial projections of the spin and summation over all final projections of the nuclear spin, as is clearly seen in formula (23). Unfortunately, nothing is said about invariability of the final state of the nucleus, which is necessary for fulfillment of the coherence condition. This problem is discussed in more detail in appendix 8.1.

$$\begin{aligned} \frac{d\sigma}{d\cos\theta} &= \sum |M(\mathbf{q})|^2 = \frac{1}{2\pi} \left(\frac{G_F E_\nu}{c^2 \hbar^2} \right)^2 \\ &\times \left\{ [g_V^e F_e + g_V^0 F_N + g_V^1 F_N(\tau_z)]^2 (1 + \cos\theta) \right. \\ &+ \sum_{m_z} \left[\frac{(g_A^e)^2}{2J_e + 1} F_e^2(\sigma_z) + \frac{[g_A^0 F_N(\sigma_z) + g_A^1 F_N(Y_z)]^2}{2J_A + 1} \right] \\ &\left. \times (3 - \cos\theta) \right\}, \end{aligned} \quad (23)$$

where $g_V^0 = -\sin^2 \theta_W$, $g_V^1 = -\frac{1}{2} + \sin^2 \theta_W$, $g_A^0 = 0$, $g_A^1 = -\frac{g_A}{2}$. This formula involves both a fully coherent vector term (upper row) and two mutually incoherent (lower row) but internally coherent electron and nucleon axial-vector terms. According to the authors [4], of the simplest form are low-energy neutrino scattering cross sections “in terms of beta decay” (when $qR \ll 1$, where R is the size of the system), at $E_\nu < \alpha m_e c^2$ for the atomic system or $E_\nu < m_\pi c^2$ for the pure nuclear one.

The quantities from [4] most interesting for further discussion are the form factors of the electron and nucleon systems, which depend on the transferred momentum q and are defined in terms of the wave functions of the entire composite system

$$\begin{aligned} F_e(q) &= \int dV_e \Psi_e^+ \left(\sum_{k=1}^{Z_e} e^{i\mathbf{q}\mathbf{r}_k} \right) \Psi_e \\ &= \int d\mathbf{r} e^{i\mathbf{q}\mathbf{r}} \int dV_e \Psi_e^+ \sum_{k=1}^{Z_e} \delta(\mathbf{r} - \mathbf{r}_k) \Psi_e \\ &= \int d\mathbf{r} e^{i\mathbf{q}\mathbf{r}} n_e(\mathbf{r}) = n_e(q); \\ F_N(q) &= \int dV_A \Psi_A^+ \left(\sum_{k=1}^A e^{i\mathbf{q}\mathbf{r}_k} \right) \Psi_A \\ &= \int d\mathbf{r} e^{i\mathbf{q}\mathbf{r}} \int dV_A \Psi_A^+ \sum_{k=1}^A \delta(\mathbf{r} - \mathbf{r}_k) \Psi_A \\ &= \int d\mathbf{r} e^{i\mathbf{q}\mathbf{r}} n_A(\mathbf{r}) = n_A(q); \\ F_N(\tau_z) &= \int dV_A \Psi_A^+ \left(\sum_{k=1}^A e^{i\mathbf{q}\mathbf{r}_k} \tau_z^k \right) \Psi_A \\ &= \int d\mathbf{r} e^{i\mathbf{q}\mathbf{r}} \int dV_A \Psi_A^+ \left(\sum_{k=1}^A \tau_z^k \delta(\mathbf{r} - \mathbf{r}_k) \right) \Psi_A \\ &= \int d\mathbf{r} e^{i\mathbf{q}\mathbf{r}} \int dV_A \Psi_A^+ \left(\sum_{k=1}^{Z_A} \delta(\mathbf{r} - \mathbf{r}_k) - \sum_{k=1}^N \delta(\mathbf{r} - \mathbf{r}_k) \right) \\ &\quad \times \Psi_A = n_p(q) - n_n(q), \\ F_e(\sigma_z) &= \int dV_e \Psi_e^+ \left(\sum_{k=1}^{Z_e} e^{i\mathbf{q}\mathbf{r}_k} \sigma_z^k \right) \Psi_e \end{aligned}$$

$$\begin{aligned} &= \int d\mathbf{r} e^{i\mathbf{q}\mathbf{r}} \int dV_e \Psi_e^+ \left(\sum_{k=1}^{Z_e} \sigma_z^k \delta(\mathbf{r} - \mathbf{r}_k) \right) \Psi_e \\ &= \int d\mathbf{r} e^{i\mathbf{q}\mathbf{r}} \int dV_e \Psi_e^+ \\ &\times \left(\sum_{k=1}^{Z_e \cap (\sigma_z^k=1)} \delta(\mathbf{r} - \mathbf{r}_k) - \sum_{k=1}^{Z_e \cap (\sigma_z^k=-1)} \delta(\mathbf{r} - \mathbf{r}_k) \right) \Psi_e \\ &= n_e^\uparrow(q) - n_e^\downarrow(q), \end{aligned} \quad (24)$$

$$\begin{aligned} F_N(\sigma_z) &= \int dV_A \Psi_A^+ \left(\sum_{k=1}^A e^{i\mathbf{q}\mathbf{r}_k} \sigma_z^k \right) \Psi_A \\ &= \int d\mathbf{r} e^{i\mathbf{q}\mathbf{r}} \int dV_A \Psi_A^+ \left(\sum_{k=1}^A \sigma_z^k \delta(\mathbf{r} - \mathbf{r}_k) \right) \Psi_A \\ &= \int d\mathbf{r} e^{i\mathbf{q}\mathbf{r}} \int dV_A \Psi_A^+ \\ &\times \left(\sum_{k=1}^{A \cap (\sigma_z^k=1)} \delta(\mathbf{r} - \mathbf{r}_k) - \sum_{k=1}^{A \cap (\sigma_z^k=-1)} \delta(\mathbf{r} - \mathbf{r}_k) \right) \Psi_A \\ &= n_A^\uparrow(q) - n_A^\downarrow(q), \end{aligned}$$

$$\begin{aligned} F_N(Y_z) &= \int dV_A \Psi_A^+ \left(\sum_{k=1}^A e^{i\mathbf{q}\mathbf{r}_k} \sigma_z^k \tau_z^k \right) \Psi_A = \int d\mathbf{r} e^{i\mathbf{q}\mathbf{r}} \\ &\times \int dV_A \Psi_A^+ \left(\sum_{k=1}^A \sigma_z^k \tau_z^k \delta(\mathbf{r} - \mathbf{r}_k) \right) \Psi_A = \int d\mathbf{r} e^{i\mathbf{q}\mathbf{r}} \int dV_A \Psi_A^+ \\ &\times \left(\sum_{k=1}^Z \sigma_z^k \tau_z^k \delta(\mathbf{r} - \mathbf{r}_k) + \sum_{k=1}^N \sigma_z^k \tau_z^k \delta(\mathbf{r} - \mathbf{r}_k) \right) \Psi_A \\ &= \int d\mathbf{r} e^{i\mathbf{q}\mathbf{r}} \int dV_A \Psi_A^+ \\ &\times \left(\sum_{k=1}^{Z \cap \sigma_z^k=1} \delta(\mathbf{r} - \mathbf{r}_k) - \sum_{k=1}^{Z \cap \sigma_z^k=-1} \delta(\mathbf{r} - \mathbf{r}_k) \right) \Psi_A - \int d\mathbf{r} e^{i\mathbf{q}\mathbf{r}} \\ &\times \int dV_A \Psi_A^+ \left(\sum_{k=1}^{N \cap \sigma_z^k=1} \delta(\mathbf{r} - \mathbf{r}_k) - \sum_{k=1}^{N \cap \sigma_z^k=-1} \delta(\mathbf{r} - \mathbf{r}_k) \right) \Psi_A \\ &= [n_Z^\uparrow(q) - n_Z^\downarrow(q)] - [n_N^\uparrow(q) - n_N^\downarrow(q)]. \end{aligned}$$

Here the spin σ_z and isospin τ_z have the form of the Pauli matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. In the limit $q \rightarrow 0$ it is seen that the form factors have a simple physical meaning of the total number of atomic electrons Z_e , total number of nucleons A , proton–neutron difference $Z - N$, total electron shell spin S_z^e , total nuclear spin S_z^A , and difference of total proton and neutron spins $S_z^Z - S_z^N$

$$F_e(q) = n_e(q) \equiv Z_e \frac{n_e(q)}{n_e(0)} \rightarrow F_e(q \rightarrow 0) = Z_e,$$

$$F_N(q) = n_A(q) \equiv A \frac{n_A(q)}{n_A(0)} \rightarrow F_N(q \rightarrow 0) = A,$$

$$\begin{aligned}
F_N(\tau_z) &= n_p(q) - n_n(q) \equiv Z \frac{n_Z(q)}{n_Z(0)} \\
- N \frac{n_N(q)}{n_N(0)} &\rightarrow F_N(\tau_z)(q \rightarrow 0) = Z - N, \\
F_e(\sigma_z) &= n_e^\uparrow(q) - n_e^\downarrow(q) \equiv Z_e \frac{n_e^\uparrow(q)}{n_e^\uparrow(0)} \\
- Z_e \frac{n_e^\downarrow(q)}{n_e^\downarrow(0)} &\rightarrow F_e(\sigma_z)(q \rightarrow 0) = 2S_z^e. \\
F_N(\sigma_z) &= n_A^\uparrow(q) - n_A^\downarrow(q) \equiv A \frac{n_A^\uparrow(q)}{n_A^\uparrow(0)} \\
- A \frac{n_A^\downarrow(q)}{n_A^\downarrow(0)} &\rightarrow F_N(\sigma_z)(q \rightarrow 0) = 2S_z^A, \\
F_N(Y_z) &= Z \frac{n_Z^\uparrow(q)}{n_Z^\uparrow(0)} - Z \frac{n_Z^\downarrow(q)}{n_Z^\downarrow(0)} - N \frac{n_N^\uparrow(q)}{n_N^\uparrow(0)} \\
+ N \frac{n_N^\downarrow(q)}{n_N^\downarrow(0)} &\rightarrow F_N(Y_z)(q \rightarrow 0) = 2S_z^Z - 2S_z^N.
\end{aligned}$$

Here $S_z^{e(A,Z,N)}$ is the projection of the total spin of the electron shell (entire nucleus, all protons, and all neutrons) on the z axis.

The authors [4] write that in atomic systems, in the case of spin-orbit coupling, the summed spin of the electrons is determined by the spin of the unfilled atomic shell. In the nuclear system, due to jj coupling and nucleon pairing, the nuclear spin S^A is actually dictated by the spin of the last unpaired nucleons, and therefore $S^A \leq 1$. Thus, axial form factors make the maximum contribution to the cross section for atoms with unfilled electron shells and large spin and nuclear axial form factors are different from zero for nuclei with odd neutron and proton numbers, but their total contribution is not large [4].

From the viewpoint of coherent cross section enhancement, of greatest interest is the coherent effect from vector form factors, which is determined by the quantity

$$\begin{aligned}
K &= [g_V^e F_e + g_V^0 F_N + g_V^1 F_N(\tau_z)]^2 \\
&\sim (g_V^e Z_e + g_V^0(Z_A + N) + g_V^1(Z_A - N))^2.
\end{aligned}$$

Nuclear form factors F_N and $F_N(\tau_z)$ are constant up to $q \sim p_F \sim 200$ MeV/ c , which corresponds to $E_\nu \leq 100$ MeV [4]. The atomic factor $F_e = Z_e$ at $q \leq 1$ keV/ c and is halved at $q \sim 1.8\sqrt{Z}$ keV/ c . Therefore, its contribution is the largest for θ (in radians), when $E_\nu \sim \frac{qc}{\theta} \sim \frac{1}{\theta}$ keV [4].

Then the conclusion is made that for $\theta \sim$ several degrees there is a region $E_\nu \leq 10$ keV, for which simul-

taneous coherent scattering off electrons and nucleons is quite significant. In the case of small-angle scattering, this effect will manifest itself up to $E_\nu \leq 100$ keV. At higher energies up to $E_\nu \leq 100$ MeV only nuclear form factors will make a contribution [4].

In [4], the following expression is given for the coherent amplification factor:

$$K_\pm = \left(\frac{1}{2}(Z_A \pm Z_e) - 2 \sin^2 \theta_W (Z_A \mp Z_e) - \frac{N}{2} \right)^2.$$

Here $\pm$ reflects uncertainty in the relative sign of the electron and nucleon coupling constants. When there is no contribution from the atomic form factor one arrives at the Freedman's result

$$K_0 = \left(\frac{(Z_A - N)}{2} - 2 \sin^2 \theta_W Z_A \right)^2.$$

Numerical estimations for ^{238}U and $E_\nu \sim 10$ keV have the form

$$\begin{aligned}
K_+ &= (Z - N/2)^2 \sim 4 \times 10^2, \\
K_- &= (1.4Z + N/2)^2 \sim 4 \times 10^4, \quad K_0 \sim 8 \times 10^3.
\end{aligned}$$

Thus, for $E_\nu \sim 10$ keV for the average cross section of inelastic processes of about 2×10^{-48} cm², consideration of atomic form factors in elastic neutrino scattering can result in a cross section of 7×10^{-46} cm² to 7×10^{-44} cm² (with atomic form factors of 1.4×10^{-44} cm² neglected), which is comparable to the cross section for inelastic processes at $E_\nu \sim 1$ MeV [4].

In conclusion, the authors [4] note that one can speak about a new field of neutrino physics, neutrino optics at $E_\nu \sim 10$ keV, for which processes of elastic neutrino scattering off the atom as a whole are of major importance. In this region, the neutrino wavelength is larger than or on the order of atomic distances, and other coherent optical phenomena are definitely possible here, e.g., those related to neutrino scattering from ordered crystal systems. In this case, additional amplification factors will arise, including the number of coherently interacting atoms [4].

A similar approach was described in [33], where methods were discussed for detecting relativistic neutrinos (solar neutrinos with the energy in the MeV range or "relic" neutrinos with $m < 10^{-4}$ eV) and, as the author puts it, nonrelativistic neutrinos (e.g., galactic neutrinos with $1 \text{ eV} < m < 10^2 \text{ eV}$). For these two types of neutrino, fundamental differences in the dependence of the coherent scattering amplitude on the neutrino scattering angle, mass, and spin direction were studied. Coherent scattering both off individual nuclei and off an extensive volume of matter, and standard nonrelativistic relations between the reflectance and the refractive index were extended for the latter

case. It was pointed out that coherence can also be studied on nuclear complexes and macroscopically large fragments of matter, where it is more appropriate to describe interaction of incident waves and matter in terms of the refractive index or the average interaction potential [33].

2.4. Attempt to Understand when Coherence Ends

The first attempt to quantitatively address the question of interrelation of coherent and incoherent neutrino–nucleus scattering was made in [34]. The importance of this investigation was dictated by extending the experimental program of searching for and studying coherent neutrino–nucleus scattering in a rather wide range of neutrino energies, sources, targets, and detector characteristics (thresholds, efficiency, etc.). And it was already not evident that the coherence condition was always fulfilled.

Indeed, a universal energy scale $E_v < 50$ MeV was usually considered as a sufficient requirement for ensuring coherence in neutrino–nucleus scattering. The authors [34] set a task to quantitatively estimate when there is coherence and when there is not. They proposed a parameter α , which, according to them, controlled the degree of coherence. The value $\alpha = 1$ meant full coherence, while $\alpha = 0$ meant full incoherence. Dependence of this parameter on the neutrino energy E_v , detection threshold for the target nucleus recoil energy $T_{\min}$, and type of the nuclear target $A(Z, N)$ was investigated. In particular, it was shown that uncertainties of the nuclear structure, as well as uncertainties in determination of α , could considerably limit sensitivity of “coherent experiments” in the search and investigation of manifestations of physics beyond the Standard Model. From the practical point of view, a more or less arbitrary “still-retained-finiteness” criterion $\alpha > 0.95$ was proposed, and its corresponding maximum neutrino energies were estimated. Some more details of this work are given below.

The differential cross section for the process $\nu + A \rightarrow \nu + A$ is used in [34] in the form (see, for example, [35, 36])

$$\frac{d\sigma}{dq^2}(q^2, E_v) = \frac{1}{2} \frac{G_F^2}{4\pi} \left(1 - \frac{q^2}{4E_v^2}\right) \times [\varepsilon Z F_Z(q^2) - N F_N(q^2)]^2, \quad \frac{d\sigma}{dT} = 2M \left[\frac{d\sigma}{dq^2} \right], \quad (25)$$

where $q \equiv |\mathbf{q}|$ is the transferred momentum, $q^2 = 2MT + T^2 \approx 2MT$, since in discussions of coherence in the elastic νA process the neutrino energy E_v and the measured nuclear recoil energy T are usually much smaller than the target nucleus mass M , $\varepsilon \equiv (1 - 4\sin^2\theta_w) = 0.045$. Kinematics con-

strain the maximum recoil energy to $T_{\max} = 2E_v^2/(M + 2E_v) \approx 2E_v^2/M$. The total cross section integrated between $q_{\min}^2 = 2MT_{\min}$ and $q_{\max}^2 = 4E_v^2[M/(M + 2E_v)] \approx 4E_v^2$ depends on $E_v, T_{\min}; M, Z, N$

$$\sigma(E_v, T_{\min}; M, Z, N) = \int_{q_{\min}^2}^{q_{\max}^2} \left[\frac{d\sigma}{dq^2}(q^2, E_v) \right] dq^2. \quad (26)$$

To detect neutrinos with energy E_v , the energy threshold should be no higher than $T_{\min} < 2E_v^2/M$.

The proton $F_Z(q^2)$ and neutron $F_N(q^2)$ nuclear form factors of the nucleus $A(Z, N)$ are used by the authors in a unified form given by formula (18) from [25]. Nuclear effects characterized by deviation of form factors from unity with increasing transferred momentum turn out to be considerable for heavy nuclei.

When $q^2 \rightarrow 0$ and $F(q^2) \approx 1$, there is the full coherence, and the cross section is maximal, in particular for $T_{\min} = 0$

$$\sigma(T_{\min} = 0) = \frac{G_F^2 E_v^2}{4\pi} [\varepsilon Z - N]^2, \quad (27)$$

i.e., the experimental evidence for the full coherence is the proportionality of the value of σ to $[\varepsilon Z - N]^2$. In addition, the differential cross section at small T is also proportional to $[\varepsilon Z - N]^2$ and does not depend on energy E_v

$$\frac{d\sigma}{dT}(T \rightarrow 0) \approx \frac{G_F^2 M}{4\pi} [\varepsilon Z - N]^2. \quad (28)$$

Therefore, loss of coherence for the total cross section σ is characterized by deviation from $[\varepsilon Z - N]^2$ scaling with increasing q^2 [34].

The total amplitude $\mathcal{A}$ of the process $\nu A \rightarrow \nu A$ in [34] was written as a sum over protons and neutrons

$$\mathcal{A} = \sum_{j=1}^Z e^{i\theta_j} \mathcal{X}_j + \sum_{k=1}^N e^{i\theta_k} \mathcal{Y}_k, \quad (29)$$

where $\mathcal{X}_j(\mathcal{Y}_k)$ denotes the coupling constant, and $e^{i\theta_j}(e^{i\theta_k})$ denotes the phase for protons (neutrons). As a result, the cross section is broken into $(N + Z)^2$ terms

$$\begin{aligned}
\sigma \propto \mathcal{A}\mathcal{A}^\dagger &= \sum_{j=1}^Z \mathcal{X}_j^2 + \sum_{k=1}^N \mathcal{Y}_k^2 \\
&+ \sum_{j=l+1}^Z \sum_{l=1}^{Z-1} [e^{i(\theta_j - \theta_l)} + e^{-i(\theta_j - \theta_l)}] \mathcal{X}_j \mathcal{X}_l \\
&+ \sum_{k=m+1}^N \sum_{m=1}^{N-1} [e^{i(\theta_k - \theta_m)} + e^{-i(\theta_k - \theta_m)}] \mathcal{Y}_k \mathcal{Y}_m \\
&+ \sum_{j=1}^Z \sum_{k=1}^N [e^{i(\theta_j - \theta_k)} + e^{-i(\theta_j - \theta_k)}] \mathcal{X}_j \mathcal{Y}_k.
\end{aligned} \quad (30)$$

Next, the authors [34] assume that effects of “loss of coherence” (decoherence) between any pair of nucleons can be parametrized in the form of an average phase mismatch angle $\langle\phi\rangle \in [0, \pi/2]$. Then, one can write

$$[e^{i(\theta_j - \theta_k)} + e^{-i(\theta_j - \theta_k)}] = 2 \cos(\theta_j - \theta_k) = 2 \cos\langle\phi\rangle, \quad (31)$$

and so on for other pairs in (30) with $(j, k) \leftrightarrow (j, l)$ and (k, m) . Therefore, the degree of coherence can be quantitatively characterized using the measured parameter α defined as $\alpha \equiv \cos\langle\phi\rangle \in [0, 1]$. Then, the ratio of the cross sections on the nucleus $A(Z, N)$ and the neutron $(0, 1)$ has the form

$$\begin{aligned}
\frac{\sigma(Z, N)}{\sigma(0, 1)} &= \{\varepsilon^2 Z + N + \varepsilon^2 Z(Z-1)\alpha \\
&+ N(N-1)\alpha - 2\varepsilon ZN\alpha\} \\
&= \{Z\varepsilon^2[1 + \alpha(Z-1)] \\
&+ N[1 + \alpha(N-1)] - 2\alpha\varepsilon ZN\}.
\end{aligned} \quad (32)$$

Hence, for $\alpha = 1$ there is the full coherence and $\sigma \propto [\varepsilon Z - N]^2$, while at $\alpha = 0$ there is the full decoherence and $\sigma \propto [\varepsilon^2 Z + N]$.⁵ It is evident from comparison of formulas (32) and (25) that the decisive effect on variations in the parameter α with increasing q is produced by the form factors $F_Z(q^2)$ and $F_N(q^2)$. As should be expected, everything is determined by the q dependence of these form factors.

In [34] it is shown that already at $E_\nu = 50$ MeV (and $T_{\min} = 0$) there is a considerable degree of loss of coherence (see Table 1). As the detection threshold of the detector increases ($T_{\min} > 0$), coherence is lost even faster.

According to the conclusions in [34], combined measurement of differential cross sections in different statements and different targets allows the transition from the coherent to the incoherent scattering to be

⁵ In [34], there is another parameter used to characterize (partial) coherence, the relative cross section variation

$\xi \equiv \frac{\sigma(\alpha)}{\sigma(\alpha=1)} = \alpha + (1-\alpha) \frac{(\varepsilon^2 Z + N)}{(\varepsilon Z - N)^2}$, which linearly depends on α , and they both are unity in the case of complete coherence.

Table 1. Maximum recoil energy, lower limit $F(q^2)$, coherence factors α for $T_{\min} = 0$ at $= 50$ MeV. It is seen that α is noticeably different from 1. Borrowed from [34]

	n	Ar	Ge	Xe
$T_{\max}$, keV	4810	134	73.8	40.9
$F(q^2) >$	0.97	0.63	0.49	0.32
$\alpha >$	—	0.77	0.68	0.57

Table 2. Maximum energy for which the coherence is preserved if the parameters $F(q_{\max}^2)$ and $\alpha > 0.952$ Borrowed from [34]

Parameter > 0.95	$E_\nu^{\max}$, MeV		
	Ar	Ge	Xe
$F(q_{\max}^2) > 0.95$	17.2	14.1	11.6
$T_{\min} = 0, \alpha > 0.95$	21.1	17.4	14.3

thoroughly investigated. In particular, low-energy reactor $\bar{\nu}_e$ and solar $^8\text{B } \nu_e$ are helpful to study the region of the full coherence ($\alpha > 0.9$), while sources of intermediate-energy neutrinos (π decay at rest) make it possible to study the transition region ($0.9 > \alpha > 0.1$).

The authors [34] write that since the goals of the investigations of coherent νA scattering are largely related to the search for New Physics, which implies comparison of measured cross sections and Standard Model predictions based on formulas (25) and (26), the sensitivity of this search procedure will be appreciably limited by uncertainties in form factors that describe nuclear effects. For this reason, measurements at $F(q^2) \simeq 1$ are preferable. In view of this, Table 2 presents maximum neutrino energies E_ν at which $F(q_{\max}^2) > 0.95$. An interesting signature of New Physics can be that σ does not have scaling of the $[\varepsilon Z - N]^2$ type even in the coherence regime $\alpha \simeq 1$ [34]. Characteristics features of this deviation could shed light on the nature of New Physics. For example, according to [37], anomalous neutrino magnetic moments would make an additional contribution proportional to Z^2 in (29), while the spin-independent interaction of dark-matter particles would lead to contributions proportional to A^2 .

It is difficult to disagree with the authors [34] that sensitivity of measurements of coherent neutrino–nucleus scattering will be noticeably limited if they are carried out in the region of poorly controlled or partially fulfilled coherence condition. It is worth noting that in [34] the question of quantitative estimation of

Fig. 4. Left: the front of the incident neutrino plane wave (solid vertical line) is scattered by the nucleons located at the fixed spatial positions $\mathbf{x}_j$ and $\mathbf{x}_k$. Because of the nonzero scattering angle θ , there arises the phase (path-length) difference $\Delta\varphi = \mathbf{q}(\mathbf{x}_j - \mathbf{x}_k)$ between two fronts of the scattered neutrino plane waves (dashed lines), which results in the loss of coherence between these two (waves) fronts. Right: the neutrino is scattered by the N_k th or N_j th nucleon whose distribution in the nucleus is described by the wave function conventionally shown in the form of a Gaussoid. The outgoing neutrino wave is a superposition of waves $e^{i\mathbf{q}\mathbf{x}_k}$ weighted with the square of the wave function $|\psi_n(\mathbf{x}_1 \dots \mathbf{x}_A)|^2$.

the coherence region was addressed for the first time, and the first attempt was made to solve it.

3. CHANGE OF THE COHERENCE PARADIGM IN νA SCATTERING

Recall the concept of neutrino–nucleus scattering coherence according to Freedman [3]. As was already mentioned (from optics), two waves are considered coherent if they have an identical frequency and shape and identical phase difference. Interaction of the neutrino with the nucleus (as a composite system) is a result of neutrino scattering off individual nucleons in the nucleus (by virtue of the weakness of the weak interaction). Each event of scattering from the k th nucleon can be described by the amplitude $\mathcal{A}^k$. Further, it is assumed that these nucleons possess definite coordinates $\mathbf{x}_k$, and therefore the amplitude $\mathcal{A}^k$ acquires an additional factor $e^{i\mathbf{q}\mathbf{x}_k}$ due to translation invariance, and the total amplitude of neutrino scattering off a composite system has the form

$$\mathcal{A} = \sum_{k=1}^A \mathcal{A}^k e^{i\mathbf{q}\mathbf{x}_k}. \quad (33)$$

According to [3], all these individual amplitudes are considered coherent (summed constructively) if for any k (any nucleon) phase factors $e^{i\mathbf{q}\mathbf{x}_k}$ are almost

identical (they can be moved outside the summation sign as a common factor). This takes place when coherence condition (1) is fulfilled, $\mathbf{q}R \ll 1$, i.e., all individual phase factors are almost equal to +1 and consequently equal to one another. Figure 4, left (similar to Fig. 3), shows a fragment of the neutrino scattering off two nucleons located at some fixed points $\mathbf{x}_j$ and $\mathbf{x}_k$ that are spatially separated from each other. The nonzero neutrino scattering angle $\theta > 0$ gives rise to a phase difference between the waves going through these points in space and, as a consequence, to loss of coherence between these waves.

3.1. “New” Coherence on a Complex Quantum Object

A question arises as to whether this coherence concept remains acceptable when the assumption of strictly fixed spatial positions of nucleons is removed. Indeed, the position of a nucleon (in the nucleus) cannot be fixed because this, as a minimum, contradicts the uncertainty principle. It can only be stated that a particular nucleon is in the nucleus, i.e., the uncertainty of its spatial position does not exceed the size of the nucleus $\Delta\mathbf{x}_j < R$. The probability of the definite position of the nucleon (in the nucleus) is given by the wave function of the multiparticle system $|m\rangle = \psi_m(\mathbf{x}_1, \dots, \mathbf{x}_A)$, where the subscripts m corre-

spond to the initial or final quantum state of the nucleus as a complex composite system.

In this case, total amplitude (33) of the transition of the nucleus from the initial state $|n\rangle$ to the final state $|m\rangle$ is generalized in the form

$$\begin{aligned}\mathcal{A}_{mn} &= \langle m | \mathcal{A} | n \rangle = \sum_{k=1}^A \mathcal{A}_{mn}^k \langle m | e^{i\mathbf{q}\mathbf{x}_k} | n \rangle \\ &= \sum_{k=1}^A \mathcal{A}_{mn}^k f_{mn}^k(\mathbf{q}).\end{aligned}$$

For the elastic transition from the state $|n\rangle$ to (the same) state $|n\rangle$ one gets

$$\mathcal{A}_{nn} = \sum_{k=1}^A \mathcal{A}_{nn}^k f_{nn}^k(\mathbf{q}), \quad (34)$$

where the quantity generalizing the phase factor $e^{i\mathbf{q}\mathbf{x}_k}$,

$$\begin{aligned}f_{mn}^k(\mathbf{q}) &= \langle m | e^{i\mathbf{q}\mathbf{x}_k} | n \rangle = \int \left(\prod_{i=1}^A d\mathbf{x}_i \right) \psi_m^*(\mathbf{x}_1 \dots \mathbf{x}_A) \\ &\times e^{i\mathbf{q}\mathbf{x}_k} \psi_n(\mathbf{x}_1 \dots \mathbf{x}_A) = \langle m | e^{i\mathbf{q}\hat{\mathbf{x}}_k} | n \rangle,\end{aligned} \quad (35)$$

is a transition (from state $|n\rangle$ to state $|m\rangle$) matrix element of the translation operator $e^{i\mathbf{q}\hat{\mathbf{x}}_k}$, where $\hat{\mathbf{x}}_k$ is the operator of the quantum position of the k th nucleon. The “diagonal” (elastic) version of formula (35)

$$\begin{aligned}f_{nn}^k(\mathbf{q}) &= \int \left(\prod_{i=1}^A d\mathbf{x}_i \right) |\psi_n(\mathbf{x}_1 \dots \mathbf{x}_A)|^2 e^{i\mathbf{q}\mathbf{x}_k} \\ &= \langle n | e^{i\mathbf{q}\mathbf{x}_k} | n \rangle = \langle n | e^{i\mathbf{q}\hat{\mathbf{x}}_k} | n \rangle,\end{aligned} \quad (36)$$

defines the nuclear form factor of the nucleon and differs from the exponential factor $e^{i\mathbf{q}\mathbf{x}_k}$ of (33) in two important aspects.

First, the quantity $f_{nn}^k(\mathbf{q})$, like $f_{mn}^k(\mathbf{q})$, does not actually depend on the coordinate of the k th nucleon despite the explicit presence of the index k . In (35), integration is performed over all variables sensitive to the particular spatial position of a particular nucleon.

Indeed, for any arbitrarily taken k the quantity $f_{mn}^k(\mathbf{q})$ can be written as an integral with explicitly distinguished integration over $\mathbf{x}_k$, i.e.,

$$\begin{aligned}&\int \left(\prod_{i \neq k}^{A-1} d\mathbf{x}_i \right) \psi_m^*(\mathbf{x}_1, \dots, \mathbf{x}_k, \dots, \mathbf{x}_A) \\ &\times \psi_n(\mathbf{x}_1, \dots, \mathbf{x}_k, \dots, \mathbf{x}_A) e^{i\mathbf{q}\mathbf{x}_k} d\mathbf{x}_k \neq f(\mathbf{x}_k),\end{aligned}$$

which obviously does not depend on $\mathbf{x}_k$.

Second, $f_{nn}^k(\mathbf{q})$, like $f_{mn}^k(\mathbf{q})$, does not depend on the index k up to the difference of the proton and neutron form factors. This statement can be proved for both fermions and bosons by replacing the integration vari-

ables and reflects symmetric properties of the wave function under permutation of its arguments.⁶ Due to these properties of $f_{mn}^k(\mathbf{q})$, it can be concluded that phases of all individual amplitudes of scattering off individual nucleons entering into overall amplitude (34) are equal to one another. More exactly, they do not appear in the corresponding expressions at all (i.e., they are of no significance), and all these amplitudes are coherent for any $\mathbf{q}$, in contrast to the commonly accepted case (33).

From this conclusion it does not follow, as may seem, that the overall amplitude stops tending to zero at sufficiently large $\mathbf{q}$. This behavior of the overall amplitude is retained since all form factors $f_{nn}^k(\mathbf{q}) \rightarrow 0$ with increasing $\mathbf{q}$. Mathematically, this behavior of $f_{nn}^k(\mathbf{q})$ is caused by the strongly oscillating factor $e^{i\mathbf{q}\mathbf{x}_k}$ that zeros out the integrand function in the integral (35). The physical cause is that there occurs incoherent (that is, with absolutely arbitrary relative phases) summation of waves belonging to the wave function of individual nucleons that are (sometimes already) outside the target nucleus. Other arguments will be presented in Section 3.

It may seem that the above statement is in conflict with the wave function corresponding to the location of the nucleons at the fixed spatial positions. Indeed, if we put

$$|\psi_n(\mathbf{y}_1 \dots \mathbf{y}_A)|^2 \propto \prod_i \delta^3(\mathbf{y}_i - \mathbf{x}_i), \quad (37)$$

where $\mathbf{y}_i$ are the variables, and $\mathbf{x}_i$ are the function parameters, expression (36) is reduced to (33), in which each term has its own phase, which explicitly contradicts our statement. This contradiction arises from that assumption (37) violates the identity principle of identical particles which requires that the wave function of a multiparticle system be either symmetric (bosons) or antisymmetric (fermions) after permutation of its arguments.

As a result, one cannot state that, for example, the i th particle (in a composite system) is located at the point $\mathbf{x}_i$, even if it is known that all particles occupy certain fixed spatial positions in the composite system. This i th particle can occupy any possible position

⁶ Indeed, if in, say, the two-nucleon case there is symmetry $\psi_n^{(*)}(\mathbf{y}, \mathbf{x}) = (\pm) \psi_n^{(*)}(\mathbf{x}, \mathbf{y})$, then $f_{mn}^{k=1}(\mathbf{q}) \equiv \int d\mathbf{x}_1 d\mathbf{x}_2 \psi_m^*(\mathbf{x}_1, \mathbf{x}_2) \psi_n(\mathbf{x}_1, \mathbf{x}_2) e^{i\mathbf{q}\mathbf{x}_1}$ can be written as $\int d\mathbf{x}_1 d\mathbf{y} \psi_m^*(\mathbf{x}_1, \mathbf{y}) \psi_n(\mathbf{x}_1, \mathbf{y}) e^{i\mathbf{q}\mathbf{x}_1}$ and after designating $\mathbf{x}_1$ as $\mathbf{x}_2$ one arrives at $f_{mn}^{k=1}(\mathbf{q}) = \int d\mathbf{x}_2 d\mathbf{y} \psi_m^*(\mathbf{x}_2, \mathbf{y}) \psi_n(\mathbf{x}_2, \mathbf{y}) e^{i\mathbf{q}\mathbf{x}_2}$. Considering the abovementioned wave function symmetry condition and permutation of the integration variables, this expression takes the form $\int d\mathbf{y} d\mathbf{x}_2 \psi_m^*(\mathbf{y}, \mathbf{x}_2) \psi_n(\mathbf{y}, \mathbf{x}_2) e^{i\mathbf{q}\mathbf{x}_2}$, which is indistinguishable from the definition of $f_{mn}^{k=2}(\mathbf{q})$.

among available fixed values $\mathbf{x}_1 \dots \mathbf{x}_A$. For this reason, consideration of expression (36) with the appropriately symmetrized δ -like wave function naturally leads to expression (33) for any k in full agreement with our conclusion. This contradiction makes it possible to better understand that the form itself of (33) ignores the fundamental principle of quantum mechanics—indistinguishability of identical particles. To illustrate, Fig. 4 (right) shows the scattering pattern considering the nucleon wave function given in the Gaussian form for the sake of illustration. Summation of neutrino waves scattered off individual nucleons, which are weighted according to $|\psi_n(\mathbf{x}_1 \dots \mathbf{x}_A)|^2$, gives a unified final scattering neutrino wave. Thus, according to the aforesaid, there are no reasons to think of diagonal ($k = j$) terms in an expansion like

$$\begin{aligned} |\mathcal{A}_{nn}|^2 &= |f_{nn}(\mathbf{q})|^2 \sum_{k,j} \mathcal{A}_{nn}^k \mathcal{A}_{nn}^{j*} \\ &= |f_{nn}(\mathbf{q})|^2 \left(\sum_k |\mathcal{A}_{nn}^k|^2 + \sum_{k \neq j} \mathcal{A}_{nn}^k \mathcal{A}_{nn}^{j*} \right) \end{aligned}$$

as resulting from incoherent (that is, “multiphase”) neutrino–nucleon interaction. Both terms, diagonal (first) and nondiagonal (second), make an identical contribution to $|\mathcal{A}_{nn}|^2$ with identical dependence on the transferred momentum $\mathbf{q}$.

It should be stressed that the traditional (Freedman’s) coherent effect in neutrino scattering off a composite system is better to refer to—within the new approach—as the effect of constructive interference of all individual (and always coherent) amplitudes of neutrino scattering off individual constituents of the composite system.

This brings up the question: What should then be considered an INcoherent (according to Freedman) interaction? The answer is that this incoherent interaction will be defined by processes in which the internal quantum state of a nucleus changes, i.e., where $n \neq m$.

3.2. What is Coherent and What is Incoherent?

The qualitative reasoning given below answers this question. For details see Subsection 10.5. For the sake of simplicity and clearness, we omit dependence of individual amplitudes on the spin, type of nucleon, and index k , that is, we assume that $\mathcal{A}_{mn}^k \rightarrow \mathcal{A}_0$. Let the nucleus be initially in the quantum state n .

Next, if it is experimentally impossible to distinguish the particular final quantum state of the nucleus m , then all possible final states of the nucleus should be considered, i.e., summation should be taken over them all. We emphasize that this point is of crucial importance for the further discussion. Unlike the case in almost all previous works where it was explicitly or implicitly assumed that the final state coincided with

the initial state (see discussion in Section 8), here, all other (i.e., inelastic) possibilities, even if minimum in number, are equally allowed. That is why it is possible to describe a smooth transition from the coherent to the incoherent regime of neutrino–nucleus scattering.

As a result of summing over all allowed final states of the nucleus, we obtain the square of the overall scattering amplitude

$$|\mathcal{A}|^2 = \sum_m |\mathcal{A}_{mn}|^2 = |\mathcal{A}_0|^2 \sum_{k,j} \sum_m f_{mn}^k f_{mn}^{j*}. \quad (38)$$

Using the definition of the matrix element (46) and the nuclear unitary operator

$$\sum_m |m\rangle \langle m| = \hat{I}, \quad (39)$$

we can recast (38) as⁷

$$\begin{aligned} |\mathcal{A}|^2 &= |\mathcal{A}_0|^2 \sum_{k,j} \langle n | e^{-iq\hat{\mathbf{x}}_j} \sum_m |m\rangle \langle m| e^{iq\hat{\mathbf{x}}_k} |n\rangle \\ &= |\mathcal{A}_0|^2 \sum_{k,j} \langle n | e^{-iq\hat{\mathbf{x}}_j} e^{iq\hat{\mathbf{x}}_k} |n\rangle. \end{aligned} \quad (40)$$

If in this formula $k = j$, the neutrino–nucleus interaction takes place on the same nucleon (Fig. 5, left), and then $\langle n | e^{-iq\hat{\mathbf{x}}_j} e^{iq\hat{\mathbf{x}}_k} |n\rangle = 1$. If $k \neq j$, the neutrino–nucleus interaction occurs on two different nucleons (Fig. 5, right), and it is convenient to introduce a two-particle correlation function

$$G_{nn}(\mathbf{q}) \equiv G(\mathbf{q}) = \langle n | e^{-iq\hat{\mathbf{x}}_j} e^{iq\hat{\mathbf{x}}_k} |n\rangle, \quad (41)$$

where $G(\mathbf{q}) = 1$ if $k = j$, and if $k \neq j$, $G(\mathbf{q})$ (like $f_{mn}^k(\mathbf{q})$) does not depend on k, j , but it is important that k and j are different.

Combining (40), (35), and (41), one can arrive at⁸

$$\begin{aligned} |\mathcal{A}|^2 &= |\mathcal{A}_0|^2 [A + G(\mathbf{q})A(A-1)] \\ &= |\mathcal{A}_0|^2 [A^2 G(\mathbf{q}) + A(1 - G(\mathbf{q}))], \end{aligned} \quad (42)$$

where A is the number of nucleons. Thus, the square of the overall amplitude $|\mathcal{A}|^2$ in (42) involves two terms, one quadratic in A and proportional to $G(\mathbf{q})$ and the other linearly dependent on A and proportional to the factor $[1 - G(\mathbf{q})]$. It is already seen that the presence of these two terms in one formula allows a

⁷ $|\mathcal{A}|^2 = \sum_m |\mathcal{A}_{mn}|^2 = |\mathcal{A}_0|^2 \sum_{k,j} \sum_m f_{mn}^{j*}(\mathbf{q}) f_{mn}^k(\mathbf{q}) = |\mathcal{A}_0|^2 \sum_{k,j} \langle n | e^{-iq\hat{\mathbf{x}}_j} \sum_m |m\rangle \langle m| e^{iq\hat{\mathbf{x}}_k} |n\rangle \rightarrow (40)$.

⁸ Indeed, the quantity $|\mathcal{A}|^2$ divided for simplicity by $|\mathcal{A}_0|^2$ can be written as a sum of two terms $\sum_{k=j}^A \langle n | e^{-iq\hat{\mathbf{x}}_j} e^{iq\hat{\mathbf{x}}_k} |n\rangle + \sum_{j,k \neq j}^A \langle n | e^{-iq\hat{\mathbf{x}}_j} e^{iq\hat{\mathbf{x}}_k} |n\rangle$, the first of which is $\sum_{k=j}^A 1$, and the second is $G(\mathbf{q}) \sum_{j,k \neq j}^A 1$, and this results in $A + A(A-1)G(\mathbf{q})$. Hence follows expression (42).

Fig. 5. Left: elastic neutrino–nucleus scattering proceeds via interaction of the neutrino with the same, k th, nucleon of the nucleus. Right: this scattering proceeds via interaction of the neutrino with different, k th and j th, nucleons of the nucleus.

smooth transition between the coherent (according to Freedman, i.e., proportional to A^2) and incoherent (i.e., proportional to A) regimes of neutrino–nucleus scattering. If the nuclear multiparticle wave function is a product of single-particle wave functions, $G(\mathbf{q})$ can be given in the form $|F(\mathbf{q})|^2$, where $F(\mathbf{q})$ is the single-particle nucleon nuclear form factor.

The form of (42) does not allow immediately perceiving the sources of the quadratic and A -linear terms. Let us show that in full compliance with the general ideas the coherent and incoherent terms emerge due to the processes in which the nucleus either remains in the initial quantum state ($n = m$) the final quantum state of the nucleus changes ($n \neq m$). To this end, we recast the expression for the overall amplitude (53) in the form of two terms

$$\begin{aligned} |\mathcal{A}|^2 &= |\mathcal{A}_0|^2 \sum_m \sum_{k,j} \langle n | e^{-iq\hat{\mathbf{x}}_j} | m \rangle \langle m | e^{iq\hat{\mathbf{x}}_k} | n \rangle \\ &= |\mathcal{A}_0|^2 \sum_{k,j} \langle n | e^{-iq\hat{\mathbf{x}}_j} | n \rangle \langle n | e^{iq\hat{\mathbf{x}}_k} | n \rangle \\ &+ |\mathcal{A}_0|^2 \sum_{k,j} \sum_{m \neq n} \langle n | e^{-iq\hat{\mathbf{x}}_j} | m \rangle \langle m | e^{iq\hat{\mathbf{x}}_k} | n \rangle, \end{aligned} \quad (43)$$

where the first term corresponds to $m = n$, and the second one corresponds to $m \neq n$. Since the square of the form factor can be written as

$$|F_n(\mathbf{q})|^2 = \langle n | e^{-iq\hat{\mathbf{x}}_j} | n \rangle \langle n | e^{iq\hat{\mathbf{x}}_k} | n \rangle,$$

the first term in (43), which then has the form $|\mathcal{A}_0|^2 \sum_{k,j} |F_n(\mathbf{q})|^2$, is obviously

$$|\mathcal{A}_0|^2 A^2 |F(\mathbf{q})|^2, \quad (44)$$

and corresponds to the coherent ($\propto A^2$) term in (42).

The second term in (43) can be written, using the definition of the covariance of quantum operators

$$\begin{aligned} \text{cov}(e^{-iq\hat{\mathbf{x}}_j}, e^{iq\hat{\mathbf{x}}_k}) &\equiv \langle n | e^{-iq\hat{\mathbf{x}}_j} e^{iq\hat{\mathbf{x}}_k} | n \rangle \\ &- \langle n | e^{-iq\hat{\mathbf{x}}_j} | n \rangle \langle n | e^{iq\hat{\mathbf{x}}_k} | n \rangle, \end{aligned} \quad (45)$$

as a sum of covariances (with $|\mathcal{A}_0|^2$ temporarily omitted)

$$\begin{aligned} &\sum_{k,j} \sum_{m \neq n} \langle n | e^{-iq\hat{\mathbf{x}}_j} | m \rangle \langle m | e^{iq\hat{\mathbf{x}}_k} | n \rangle \\ &= \sum_{k,j} \left(\sum_m \langle n | e^{-iq\hat{\mathbf{x}}_j} | m \rangle \langle m | e^{iq\hat{\mathbf{x}}_k} | n \rangle \right. \\ &\quad \left. - \sum_{m=n} \langle n | e^{-iq\hat{\mathbf{x}}_j} | m \rangle \langle m | e^{iq\hat{\mathbf{x}}_k} | n \rangle \right) \\ &= \sum_{k,j} (\langle n | e^{-iq\hat{\mathbf{x}}_j} e^{iq\hat{\mathbf{x}}_k} | n \rangle - \langle n | e^{-iq\hat{\mathbf{x}}_j} | n \rangle \langle n | e^{iq\hat{\mathbf{x}}_k} | n \rangle) \\ &\equiv \sum_{k,j} \text{cov}(e^{-iq\hat{\mathbf{x}}_j}, e^{iq\hat{\mathbf{x}}_k}). \end{aligned}$$

Next, considering that at $k = j$ the covariant term turns into an “addition to the form factor”

$$\begin{aligned} \text{cov}(e^{-iq\hat{\mathbf{x}}_k}, e^{iq\hat{\mathbf{x}}_k}) &= \langle n | e^{-iq\hat{\mathbf{x}}_k} e^{iq\hat{\mathbf{x}}_k} | n \rangle \\ &- \langle n | e^{-iq\hat{\mathbf{x}}_k} | n \rangle \langle n | e^{iq\hat{\mathbf{x}}_k} | n \rangle = 1 - |F_n(\mathbf{q})|^2, \end{aligned}$$

this sum of covariances can be transformed by explicitly separating out the summation over the equal indices $k = j$

$$\begin{aligned} \sum_{k,j} \text{cov}(e^{-iq\hat{\mathbf{x}}_j}, e^{iq\hat{\mathbf{x}}_k}) &= \sum_{k=j} \text{cov}(e^{-iq\hat{\mathbf{x}}_j}, e^{iq\hat{\mathbf{x}}_j}) \\ &+ \sum_{k \neq j} \text{cov}(e^{-iq\hat{\mathbf{x}}_j}, e^{iq\hat{\mathbf{x}}_k}) \\ &= \sum_{k=j} (1 - |F_n(\mathbf{q})|^2) + \sum_{k \neq j} \text{cov}(e^{-iq\hat{\mathbf{x}}_j}, e^{iq\hat{\mathbf{x}}_k}). \end{aligned}$$

As a result, the second term in (43) can be written as

$$|\mathcal{A}_0|^2 \left[A(1 - |F(\mathbf{q})|^2) + \sum_{k \neq j} \text{cov}(e^{-iq\hat{\mathbf{x}}_j}, e^{iq\hat{\mathbf{x}}_k}) \right].$$

The covariant terms involved in this expression are identically equal to zero when the wave function of the multiparticle system is a product of single-particle wave functions, and in this particular case (which, however, illustrates the point well) the second term in (43) becomes equal to the incoherent ($\propto A$) term in (42)

$$|\mathcal{A}_0|^2 A(1 - |F(\mathbf{q})|^2). \quad (46)$$

Thus, the above qualitative consideration allows us to conclude that (i) the coherent (when the nuclear

Fig. 6. Neutrino interaction with the nucleon with the momentum $\mathbf{p} = \alpha\mathbf{q}$ equal to (48). Since the nucleus was at rest, all other nucleons had the momentum $-\mathbf{p}$ (dashes). The momentum $\mathbf{p} + \mathbf{q} = (1 + \alpha)\mathbf{q}$ gained by the nucleon is directed along $\mathbf{q}$. In the figure, $\mathbf{p}$ and $\mathbf{p} + \mathbf{q}$ are slightly diverged (for clarity). The energy and the momentum $\mathbf{q}$ acquired by the nucleon are transferred to the entire nucleus without changing its quantum state.

state does not change, $n = m$ contribution to the νA cross section corresponds to elastic scattering and has the form $|\mathcal{A}_0|^2 A^2 |F(\mathbf{q})|^2$ (the first term in (43) or formula (44)) and (ii) the incoherent (the nucleus undergoes transition to another state, $n \neq m$) contribution to the νA cross section corresponds to all allowed inelastic processes and has the form $|\mathcal{A}_0|^2 A(1 - |F(\mathbf{q})|^2)$ (the second term in (43) or formula (46)). Note that the use of the condition for completeness of nuclear states (39) was of crucial importance in the above consideration.

Correspondence of elastic and inelastic processes to coherent and incoherent interactions was earlier adopted, for example, in [21, 38–40], where contributions to the cross sections for neutrino–nucleus scattering were numerically calculated on the basis of nuclear-physics models.

3.3. Kinematic Paradox of Traditional Coherence

It is universally adopted that coherent enhancement of the probability for the incident particle (neutrino) scattering off a complex composite system (nucleus) corresponds to the case where this system remains after interaction in its initial quantum state. However, this statement, obvious from the viewpoint of quantum mechanics basics, leads on closer examination to a kinematic paradox that, as far as we know, has never been discussed in the literature.

Indeed, if the nucleus is treated as a complex system of nucleons with which neutrinos interact exclusively, both the nucleon and the nucleus (containing this nucleon) acquire the identical momentum $\mathbf{q}$ transferred to them by the neutrino. The neutrino “donated” this momentum (in the interaction event) and moved further away, being “not interested” in who got it, how it was redistributed, and what ever became of the nucleus.

If both the nucleus and this (active) nucleon are supposed to be at rest before the interaction with the neutrino, it turns out that after the interaction with the

neutrino the kinetic energy of the nucleon $T_N = \mathbf{q}^2/2m_N$ is larger than the kinetic energy of the nucleus $T_A = \mathbf{q}^2/2m_A$ by m_A/m_N .

However, since it is believed that the nucleus remains in the same quantum state with the same internal energy (conservation of the nuclear energy and integrity of a nucleus), T_N and T_A should be equal

to the same lost neutrino energy $E_\nu - E'_\nu$, i.e., these quantities should be equal to each other. It is physically clear that the momentum acquired by the nucleon is redistributed among the other nucleons (since the nucleus retains its integrity) and thus the entire nucleus acquires the total momentum obviously equal to $\mathbf{q}$.

Nevertheless, this kinematic paradox arises from the assumption that the nucleon “accepting the interaction” is at rest. This assumption is usually quite applicable when the momentum of the internal motion of the nucleon in the nucleus (Fermi momentum) $\mathbf{p}$ is much smaller than the mass of the nucleon m_N . However, here is a kinematic controversy: the requirement of nuclear integrity and the requirement that there be $T_N = T_A$ cannot be simultaneously satisfied at any momentum $\mathbf{p}$ of the internal motion of the nucleon in the nucleus.

Assuming that $T_N = [E_\nu - E'_\nu] = T_A$ and using the energy conservation law (before and after interaction) in the form

$$\frac{(\mathbf{p} + \mathbf{q})^2}{2m_N} - \frac{\mathbf{p}^2}{2m_N} = \frac{\mathbf{q}^2}{2m_A}, \quad (47)$$

one can find appropriate values for the momentum $\mathbf{p}$ of the active nucleon that got involved in the interaction with the neutrino. It turns out to be

$$\mathbf{p} = -\frac{\mathbf{q}}{2} \left(1 - \frac{m_N}{m_A} \right). \quad (48)$$

Thus, conservation of the momentum and requirement of invariability of the quantum state of the nucleus after interaction give a qualitative picture of coherent neutrino–nucleus interaction in the form symbolically presented in Fig. 6. From this picture (Fig. 6) there follow a few important observations, which were not, to our knowledge, discussed in the literature, at least in the context of coherent neutrino scattering off a bound composite system (a nucleus).

First, not any nucleon can interact with the neutrino in such a way that after the interaction of the neutrino with the nucleus the latter remains in the initial (unchanged) quantum state; only those can that “have the luck” to have the momentum (in the nucleus before interaction) defined by formula (48). It should be stressed that in this discussion the word “nucleon,” which usually means simply a constituent of a nucleus, gets a connotation (or true meaning) in

Fig. 7. Longitudinal component p_L of the internal momentum of the (active) nucleon corresponding to the energy–momentum conservation law in the neutrino–nucleus scattering as a solution of equation (50) at different nuclear excitation energies $\Delta\epsilon_{mn}$.

the sense that this proton or neutron interacts with the neutrino and at the same time “feels” the effect of other nucleons of the nucleus as a bound composite system. In other words, for the nucleus to retain its self-identity after the interaction that results in its acquiring the momentum $\mathbf{q}$, this nucleus “should sort of prepare inside” a nucleon with internal momentum (48). Only then the (elastic) coherent scattering, in which the nucleus retains its initial quantum state, becomes possible.

Second, since the distribution of nucleon momenta in the nucleus is governed by the multiparticle wave function of the bound system (nucleus) localized in the configuration and momentum spaces, large nucleon momenta (in the nucleus) are usually much less probable than small ones. It is thus clear why the coherent amplification factor works poorly for larger transferred momenta $\mathbf{q}$ (there can rarely be an appropriate nucleon momentum), while for small $\mathbf{q}$ the probability to find a nucleon with an appropriate momentum in the nucleus is considerably higher (cf. condition (1)). Formally (mathematically), this effect depends on the square of the nuclear form factor $|F(\mathbf{q})|^2$.

This discussion can be extended to the incoherent neutrino–nucleus scattering, in which the nucleus changes its internal quantum state $|n\rangle \rightarrow |m\rangle$, where $n \neq m$. Condition (47) should be generalized by considering the nonzero difference of the energy levels of the nucleus $\Delta\epsilon_{mn} \neq 0$

$$\begin{aligned} E_{\mathbf{p}+\mathbf{q}} - E_{\mathbf{p}} &= \frac{\mathbf{q}^2}{2m_A} + \Delta\epsilon_{mn} \rightarrow \sqrt{E_{\mathbf{p}}^2 + 2\mathbf{p}\mathbf{q} + \mathbf{q}^2} \\ &- E_{\mathbf{p}} = \frac{\mathbf{q}^2}{2m_A} + \Delta\epsilon, \end{aligned} \quad (49)$$

where $E_{\mathbf{p}} = \sqrt{m_N^2 + \mathbf{p}^2}$. Expression (49) involves relativistic formulas for the nucleon energy, since for small $\mathbf{q}$ the momentum $\mathbf{p}$, solution of (49), can be comparable to the nucleon mass m_N .

With $\mathbf{p}$ taken in the form of a sum of the longitudinal $\mathbf{p}_L$ and the transverse $\mathbf{p}_T$ momentum, the exact solution of (49) for the fixed momentum transfer $\mathbf{q}$ has the form⁹

$$p_L = -\frac{|\mathbf{q}|}{2} \left(1 - \sqrt{\beta} \sqrt{1 + \frac{4m_{N,T}^2}{\mathbf{q}^2(1-\beta)}} \right), \quad (50)$$

where $\beta = \frac{E_{mn}^2}{\mathbf{q}^2}$, $E_{mn} = \frac{\mathbf{q}^2}{2m_A} + \Delta\epsilon_{mn}$, $m_{N,T}^2 = m_N^2 + p_T^2$.

Figure 7 shows solutions (50) as a function of $|\mathbf{q}|$.

When $\Delta\epsilon_{mn} = 0$, the solution in Fig. 7 reproduces result (48). Here β from (50) is very small $\beta = \frac{\mathbf{q}^2}{4m_A^2} = \frac{T_A}{2m_A} \ll 1$, and $p_L \approx -\frac{|\mathbf{q}|}{2} \left(1 - \frac{m_{N,T}}{m_A} \right)$ coincides with (48) at $p_T = 0$.

Thus, it can be seen in Fig. 7 that the longitudinal component of the momentum of the “active” nucleon capable of participating in the coherent neutrino–nucleus scattering is always directed against the three-momentum $\mathbf{q}$ transferred to the nucleus. If there is inelasticity with $\Delta\epsilon_{mn} \neq 0$, the solution of (49) is appreciably different in the region of small transferred momenta $\mathbf{q}$. Several different cases are possible.

⁹ Equation (47) can be solved analytically using the MATHEMATICA package or manually by temporarily designating $E \equiv \sqrt{m^2 + p^2 + 2pq + q^2} - \sqrt{m^2 + p^2}$, which yields $m^2 + p^2 + 2pq + q^2 = (E + \sqrt{m^2 + p^2})^2 = E^2 + 2E\sqrt{m^2 + p^2} + m^2 + p^2$, and hence, cancelling identical terms on the right and on the left, we have $2pq + q^2 = E^2 + 2E\sqrt{m^2 + p^2}$ or $2pq + d = 2E\sqrt{m^2 + p^2}$, where the designation $d = (q^2 - E^2)$ is introduced. Then, raising to the second power $4E^2(m^2 + p^2) = (2pq + d)^2 = 4p^2q^2 + 4dpq + d^2$ and transforming the resulting expression to $4p^2(E^2 - q^2) - 4dpq + 4E^2m^2 - d^2 = 0$, we arrive at the quadratic equation $4dp^2 + 4dpq + d^2 - 4E^2m^2 = 0$. That is, we have the equation $p^2 + pq + b = 0$, where $b = \frac{d^2 - 4E^2m^2}{4d}$, and two solutions

$$p_{1,2} = -\frac{q}{2} \left(1 \pm \left[\frac{q^2 - 4b}{q^2} \right]^{1/2} \right). \text{ Further, } q^2 - 4b = E^2 + \frac{4E^2m^2}{q^2 - E^2} = q^2\beta \left(1 + \frac{4m^2}{q^2} \frac{1}{1-\beta} \right), \text{ where } \beta = \frac{E^2}{q^2} \equiv \frac{1}{q^2} \left(\frac{\mathbf{q}^2}{2m_A} + \Delta\epsilon_{mn} \right)^2. \text{ Finally, one physical solution has the form } p = -\frac{q}{2} \left(1 - \sqrt{\beta} \sqrt{1 + \frac{4m^2}{q^2} \frac{1}{1-\beta}} \right).$$

Fig. 8. Both balls are initially at rest. Then the right ball acquires momentum $\mathbf{q}$ (left). After some time, their center of mass moves with momentum $\mathbf{q}$, and half of the energy gained from outside is accumulated in the potential energy of the spring. It is compressed and extended as shown at the right of this figure.

At $\mathbf{q} = 0$ Eq. (49) has no solution, which simply corresponds to impossibility of excitation of the nucleus at zero momentum transfer to it. Next, at small $|\mathbf{q}|$, slightly larger than the minimum possible

value $|\mathbf{q}_{\min}| = \Delta\epsilon_{mn} + \frac{\Delta\epsilon_{mn}^2}{2m_A}$, the solution in the form of

(50) diverge, i.e., $p_L \rightarrow \infty$, and therefore the nonrelativistic approximation in (49) is inapplicable at small $\mathbf{q}$. The probability of finding an active (in terms of participation in scattering) nucleon with this internal momentum in the nucleus is negligible. Therefore, for small $|\mathbf{q}|$ incoherent (more precisely, inelastic) scattering is almost impossible, while for coherent (more precisely, elastic) scattering there are no restrictions. As $|\mathbf{q}|$ increases, the nuclear transition $|n\rangle \rightarrow |m\rangle$ with the difference of nuclear energy levels $\Delta\epsilon_{mn}$ becomes possible with the minimum degree of suppression corresponding to the internal momentum of the active nucleon $\mathbf{p} \approx 0$. The situation is again entirely opposite for coherent (elastic) scattering. The above kinematic consideration allows the dependence of the neutrino–nucleus scattering on the transferred momentum $\mathbf{q}$ to be qualitatively understood, but, of course, does not give the full picture. At the transition $|n\rangle \rightarrow |m\rangle$ ($n \neq m$), the matrix element $\langle m | e^{i\mathbf{q}\hat{\mathbf{x}}} | n \rangle$, where $\hat{\mathbf{x}}$ is the quantum position operator, actually governs this functional dependence. This issue is quantitatively considered in appendix 10.2.

Mechanical analogy of elastic and inelastic scattering

The kinematic paradox of traditional coherence discussed above has a simple mechanical analog. To illustrate, we consider a mechanical system consisting of two balls with identical mass m connected by a massless elastic spring.

In the first case, let both balls be at rest, and let one of them (e.g., the right one) acquire (due to the impact from outside) a momentum $\mathbf{q}$ and a corresponding

Fig. 9. Balls execute oscillatory motion with the maximum momentum $\pm\mathbf{q}/4$. The center of mass is at rest. At the time when $-\mathbf{q}/4$ the right ball additionally acquires momentum $\mathbf{q}$ (left). Then the center of mass moves with momentum $\mathbf{q}$, and the spring has the initial potential energy, i.e., the system preserved its internal state unchanged. The balls continue moving with the maximum momentum $\pm\mathbf{q}/4$.

energy $\mathbf{q}^2/2m$ (Fig. 8, left). After redistribution of this momentum between the balls, their center of mass moves with the momentum $\mathbf{q}$ (momentum conservation law), but their total kinetic energy is half as large, $\mathbf{q}^2/4m$, since their mass is $2m$. The remaining half of the kinetic energy of the right ball (Fig. 8, right) passed into the potential energy of the spring. In this case, which is an analog of inelastic interaction, the initial ground state of the system (spring at rest) changed to an excited state (spring has internal energy), and because of this, the balls execute oscillatory motion around the common center of mass (Fig. 8, right).

In the second case, the balls (due to spring compression and extension) execute periodic motion toward each other and back, while their center of mass remains fixed. The momenta of the balls at any time instant are equal in magnitude and opposite in direction. The system is in an excited state. Let the maximum momentum of the ball, when the potential energy of the spring is zero, be $\pm\mathbf{q}/4$ (Fig. 9, left). One of these momenta is chosen to be $-\mathbf{q}/4$ according to (48). The total energy of two balls is $(\mathbf{q}/4)^2/2m + (\mathbf{q}/4)^2/2m = \mathbf{q}^2/16m$. Immediately after the right ball (Fig. 9) with its momentum $-\mathbf{q}/4$ acquired an additional momentum $\mathbf{q}$, the momenta and the energies of the balls became $(+\mathbf{q}/4, +3/4\mathbf{q})$ and $(\mathbf{q}^2/32m, 9\mathbf{q}^2/32m)$. The total energy of the system is now $5\mathbf{q}^2/16m$. Since the center-of-mass energy is $\mathbf{q}^2/4m$, the potential energy stored in the spring does not change and is $\mathbf{q}^2/16m$. That is, after the interaction this system (of two balls connected by a spring) remains in its initial state with the same internal potential energy (of the spring). The change in the kinetic energy of the impacted ball exactly corresponds to the acquired kinetic energy of the center of mass of the system. This is an example of elastic scattering.

3.4. Summary on the Change in the Coherent Scattering Paradigm

(1) When a light beam is diffracted, the position of the scattering centers is really strictly fixed, i.e., the diffraction grating is a macroscopic object with geometric characteristics (position and size of slits etc.) well-defined in space. However, for a nucleus as a microscopic quantum (complex and bound) system, it is in principle impossible to exactly fix positions of scatterers, and strictly speaking, this logic is improper.

(2) Not any nucleon can interact with the neutrino in such a way that after the interaction of the neutrino with the nucleus the latter remains in the initial (unchanged) quantum state. Only those can that “have the luck” to have the momentum (in the nucleus before interaction) defined by formula (48). In other words, for the nucleus to retain its self-identity after the interaction that results in its acquiring the momentum $\mathbf{q}$, this nucleus “should sort of prepare inside” a nucleon with internal momentum (48). Only then the (elastic) coherent scattering, in which the nucleus retains its initial quantum state, becomes possible.

(3) The coherent (when the nuclear state does not change), or proportional to A^2 , which is also terminologically equivalent in the literature, contribution to the cross section corresponds to the elastic scattering and has the form $|\mathcal{A}_0|^2 A^2 |F(\mathbf{q})|^2$. The incoherent (when the nucleus goes to another state), or proportional to A , which is terminologically equivalent, contribution to the cross section corresponds to all allowed inelastic processes and has the form $|\mathcal{A}_0|^2 A(1 - |F(\mathbf{q})|^2)$.

(4) What is traditionally called a coherent effect in the neutrino scattering off a composite system should be better called the effect of constructive interference of all individual (always coherent) amplitudes of neutrino scattering off individual elements of the composite system. For (anti)neutrinos with energies in the region of tens of MeV both terms of the cross section are coherent in the sense that all nucleons take part in the scattering at the level of scattering amplitudes (and not cross sections). Therefore, strictly speaking, in this kinematic region it is more correct to speak of elastic and inelastic scattering rather than coherent and incoherent [20].

4. KINEMATICS OF ELASTIC AND INELASTIC νA SCATTERING

Let us denote the four-momenta of the incident and outgoing (anti)neutrinos by $k = (E_\nu, \mathbf{k})$ and $k' = (E_\nu', \mathbf{k}')$ and the four-momenta of the initial and final nuclear states by P_n and P_m' (see Fig. 2, left). If the

(anti)neutrino impinges on the nucleus at rest along the z axis, we can write

$$\begin{aligned} k &= (E_\nu, 0, 0, k_z = E_\nu), \\ k' &= (E_\nu', k'_x = E_\nu' \sin \theta, 0, k'_z = E_\nu' \cos \theta), \\ P_n &= (P_n^0 = m_A + \varepsilon_n, 0, 0, 0), \\ P_m' &= (P_m'^0 = \varepsilon_m + \sqrt{m_A^2 + (P_m^x)^2 + (P_m^z)^2}, P_m^x, 0, P_m^z), \end{aligned}$$

where ε_m is the excitation energy of the state m .

The total energy of the nuclear state $|P_n\rangle$ is $P_n^0 = E_p + \varepsilon_n$, where ε_n is the internal energy of the given nuclear quantum state. In the laboratory coordinate system (where nucleus is at rest), the energy $E_\nu' \equiv k'_0$ of the (anti)neutrino scattered off the nucleus with mass m_A depends on the angle θ between the vectors $\mathbf{k}$ and $\mathbf{k}'$

$$E_\nu' = \frac{m_A(E_\nu - \Delta\varepsilon_{mn}) - E_\nu \Delta\varepsilon_{mn} + \Delta\varepsilon_{mn}^2/2}{m_A + E_\nu(1 - \cos \theta) - \Delta\varepsilon_{mn}}, \quad (51)$$

where the difference of the energies of the nuclear states $|m\rangle$ and $|n\rangle$ is designated as

$$\Delta\varepsilon_{mn} = \varepsilon_m - \varepsilon_n. \quad (52)$$

If the (anti)neutrino is taken to be massless, $m_\nu^2 = 0$, the four-momentum transferred to the nucleus $q = (q_0, \mathbf{q}) \equiv k - k'$ is related to the kinetic energy of the recoil nucleus T_A and other kinematic variables as follows:

$$\begin{aligned} q^2 &= (k - k')^2 = -2kk' = -2(E_\nu E_\nu' - |\mathbf{k}||\mathbf{k}'| \cos \theta) \\ &= -2E_\nu E_\nu' (1 - \cos \theta), \end{aligned}$$

$$q_0 = E_\nu - E_\nu' = P_m^0 - P_n^0 = \Delta\varepsilon_{mn} + T_A,$$

$$\mathbf{q}^2 = (\mathbf{k} - \mathbf{k}')^2 = E_\nu^2 + (E_\nu')^2 - 2E_\nu E_\nu' \cos \theta \approx 2m_A T_A,$$

$$|\mathbf{q}| = \left(E_\nu^2 + E_\nu'^2 - 2E_\nu E_\nu' \cos \theta \right)^{1/2} \approx (2m_A T_A)^{1/2}.$$

In the center-of-mass system of the (anti)neutrino and the nucleus (variables are marked with $*$) the expression for the transferred momentum is

$$q^2 = -4E_{\nu,n}^* E_{\nu,m}^* \sin^2 \frac{\theta^*}{2}, \quad (53)$$

where $E_{\nu,n}^* = \frac{s_{A,n} - m_{A,n}^2}{2\sqrt{s_A}}$ is the energy of the (anti)neutrino scattered off the nuclear state $|n\rangle$, in which $s_{A,n} = (k + P_n)^2$ is the square of the total (initial) energy and $m_{A,n} = m_A + \varepsilon_n$ can be arbitrarily called the excited nuclear mass involving both the internal excitation energy of the nucleus and the internal energy stored in its mass.

The minimum and maximum values of the transferred momentum q^2 corresponding to $\sin^2 \frac{\theta^*}{2} = 1$ and 0 are

$$q_{\min}^2 = -4E_{\nu,n}^* E_{\nu,m}^* = \frac{-4E_{\nu}^2}{\sqrt{\left(1 + \frac{2E_{\nu}}{m_{A,n}}\right)\left(1 + \frac{2E_{\nu}}{m_{A,m}}\right)}}, \quad (54)$$

$$q_{\max}^2 = 0.$$

For relatively massive nuclei with $\Delta\epsilon_{mn}$ of about several hundreds of keV and also when the experimentally detected signal is determined by the kinetic recoil energy of the nucleus, the transferred momentum q^2 can be approximated by the expression

$$q^2 \approx -\mathbf{q}^2 \approx -2m_A T_A. \quad (55)$$

Considering that before the interaction with the (anti)neutrino the nucleus is at rest (in the lab system), its kinetic recoil energy is

$$T_A = \sqrt{m_A^2 + \mathbf{q}^2} - m_A \quad (56)$$

or $T_A = E_{\nu} - E_{\nu}' - \Delta\epsilon_{mn}$. Hence, with E_{ν}' from (51), comes the exact formula for the kinetic (recoil) energy of the nucleus T_A

$$T_A = \frac{E_{\nu}(E_{\nu} - \Delta\epsilon_{mn})(1 - \cos \theta) + \Delta\epsilon_{mn}^2/2}{m_A + E_{\nu}(1 - \cos \theta) - \Delta\epsilon_{mn}}. \quad (57)$$

If we confine ourselves to the case (realizable with a good accuracy for coherent scattering) of $m_A \gg E_{\nu}$ (and also, obviously, $m_A \gg \Delta\epsilon_{mn}$), there is an approximate expression for the nuclear kinetic energy¹⁰

$$T_A \approx \frac{E_{\nu}(E_{\nu} - \Delta\epsilon_{mn})(1 - \cos \theta) + \Delta\epsilon_{mn}^2/2}{m_A}. \quad (58)$$

Let us consider three limiting cases of expression (58) for the kinetic recoil energy.

(i) To the forward (anti)neutrino scattering there corresponds the zero angle $\theta = 0$, or $\cos \theta = 1$, to

¹⁰ Indeed, $T_A = E_{\nu} - \frac{\Delta^2\epsilon_{mn}/2 + E_{\nu}m_A - \Delta\epsilon_{mn}m_A - \Delta\epsilon_{mn}E_{\nu}}{m_A + E_{\nu} - E_{\nu}\cos\theta - \Delta\epsilon_{mn}} - \Delta\epsilon_{mn}$. By reducing to the common denominator, removing the parentheses and cancelling out the identical terms, we obtain $T_A = \frac{E_{\nu}^2 - E_{\nu}^2\cos\theta - E_{\nu}\Delta\epsilon_{mn} + \Delta\epsilon_{mn}E_{\nu}\cos\theta + \Delta^2\epsilon_{mn}/2}{m_A + E_{\nu} - E_{\nu}\cos\theta - \Delta\epsilon_{mn}} = \frac{E_{\nu}(E_{\nu} - \Delta\epsilon_{mn})(1 - \cos\theta) + \Delta^2\epsilon_{mn}/2}{m_A + E_{\nu}(1 - \cos\theta) - \Delta\epsilon_{mn}}$. Hence, considering $m_A \gg E_{\nu}, m_A \gg \Delta\epsilon_{mn}$, there follows formula (58).

which there corresponds the minimum kinetic energy of the recoil nucleus

$$T_A^{\min} = \lim_{\cos\theta \rightarrow 1} T_A \approx \frac{\Delta\epsilon_{mn}^2}{2m_A}, \quad (59)$$

which is zero if $m = n$, since in this case neither the momentum nor the energy is transferred to the nucleus. When $m \neq n$, the energy $q_0 = \Delta\epsilon_{mn}$ and the three-momentum $\mathbf{q}$ equal in magnitude to q_0 (since $\theta = 0$) are transferred to the nucleus, which results in $T_A^{\min} = \mathbf{q}^2/2m_A$.

(ii) To the backward (anti)neutrino scattering there corresponds the angle $\theta = \pi$, or $\cos \theta = -1$, to which there corresponds the maximum nuclear kinetic recoil energy

$$T_A^{\max} = \lim_{\cos\theta \rightarrow -1} T_A \approx \frac{(2E_{\nu} - \Delta\epsilon_{mn})^2}{2m_A}. \quad (60)$$

Indeed, when $m = n$ (elastic scattering), no energy is used for changing the internal structure of (exciting) the nucleus, while the transferred three-momentum is twice as high as the initial momentum (and energy) of the (anti)neutrino (backward scattering). Therefore, $T_A^{\max} = (2E_{\nu})^2/2m_A$. When $m \neq n$, the energy equal to $\Delta\epsilon_{mn}$ is used to change the internal structure of (excitation) the nucleus, and the energy $2E_{\nu}$ (and the momentum, since $m_{\nu} = 0$), which is capable of passing into the nuclear kinetic recoil energy, should be decreased by this value, which is demonstrated by (60).

(iii) By and large, at the same E_{ν} the kinetic nuclear recoil energy is lower when the nucleus changes its quantum state ($m \neq n$) than when the nucleus remains unchanged ($m = n$). Practically, this can be considered by decreasing the (initial) neutrino energy by $\Delta\epsilon_{mn}$. This can be a considerable decrease if the incident (anti)neutrino energy E_{ν} is comparable to the nuclear excitation energy $\Delta\epsilon_{mn}$.

For heavy nuclei, such as ^{133}Cs and/or ^{127}I used as targets in the COHERENT experiment [8], energies of the first excitation levels are on the order of 100 keV. These excitation energies can be ignored, because the neutrino beam energies are of the order of several tens of MeV. Therefore, kinetic energies of recoil nuclei have the same order of magnitude for both elastic and inelastic neutrino–nucleus scattering.

Figure 10 shows possible kinetic recoil energies of ^{133}Cs nuclei as a function of the incident (anti)neutrino energy E_{ν} with allowance for the influence of the nuclear excitation energy $\Delta\epsilon_{mn}$ and the neutrino scattering angle $\cos \theta$. The top panel of Fig. 10 demonstrates strong dependence of the kinetic nuclear recoil energy T_A on the (anti)neutrino scattering angle θ .

Also, there is dependence on nonzero nuclear excitation energies $\Delta\epsilon_{mn}$ (Fig. 10, bottom panel), but this dependence is noticeably weaker than the angular dependence. The weaker dependence on nonzero $\Delta\epsilon_{mn}$ is caused by partial compensation due to the term $\Delta\epsilon_{mn}^2/2$ in the numerator of (58) at $E_\nu \approx \Delta\epsilon_{mn}$ and by almost lost dependence on $\Delta\epsilon_{mn}$ when $E_\nu \gg \Delta\epsilon_{mn}$.

5. AMPLITUDE OF (ANTI)NEUTRINO SCATTERING OFF NUCLEI

The microscopic description of the neutrino (anti-neutrino) scattering off the nucleus as a composite bound system proposed below follows from first principles of quantum field theory and rests on the fundamental process of (anti)neutrino scattering off the nucleon as a basic (structureless) constituent of the nucleus.

The initial and the final target nucleus (Fig. 2, left) will be thought of as being a Fock state $|P_l\rangle$, when the nucleus has a four-momentum P_l and is in an l th internal quantum state ($l = n, m$), being a superposition of free nucleons weighted with their common bound state wave function. The quantity $|P_l\rangle$ is a product of two wave functions describing the internal nuclear structure and the motion of the center of mass of the nucleus as a unified composite system. The internal wave function depends on $A - 1$ three-momenta, since one of the three-momenta (more explicitly, one combination of A three-momenta) is used to describe motion of the nucleus as a whole. In this connection, it is convenient to consider the Fock state $|n\rangle$ as describing the nucleus at rest in the n th quantum state. When the nucleus has a zero momentum, both $|P_n\rangle$ and $|n\rangle$ describe the same quantum number of the nucleus but differ only by the normalization factors (see formulas (65) and (64)). For details see appendix 10.1.4.

5.1. Wave Function of the Nucleus as a Bound Composite System

To continue the consideration, it is necessary to specify the nuclear wave function. We will assume (see appendix 10.1.4) that the Fock state $|P_n\rangle$ completely describing the nucleus A in its n th quantum state with the four-momentum P_n (see Fig. 2, left) is a product of states of free nucleons $|\{p\}\rangle$ weighted with their common bound state wave function $\tilde{\Psi}'_n(\{p\})$. The latter is in turn a product of the wave function $\tilde{\Psi}_n(\{p^*\})$ describing the internal structure of the nucleus in its center-of-mass system (the corresponding momenta are marked with $*$) and the wave function $\Phi_n(p)$ describing motion of the nucleus as a whole with the

Fig. 10. Kinetic (recoil) energy of the ^{133}Cs nucleus expectedly resulting from neutrino–nucleus interaction as a function of the incident neutrino energy. The top panel, corresponding to elastic scattering, when nucleus is not excited, i.e., $\Delta\epsilon_{mn} = 0$, is for different $\cos\theta$, where θ is the neutrino scattering angle. The bottom panel, corresponding to inelastic scattering, when the nucleus becomes excited, i.e., $\Delta\epsilon_{mn} \neq 0$, is for backward scattering $\cos\theta = -1$.

momentum $\mathbf{p} = \sum_{i=1}^A \mathbf{p}_i$ and the projection of the nuclear spin s

$$\tilde{\Psi}'_n(\{p\}) = \tilde{\Psi}_n(\{p^*\})\Phi_n(p), \quad \text{where } p = (\mathbf{p}, s). \quad (61)$$

Considering (61), the following equation will be used for the state $|P_n\rangle$ completely describing the nucleus A :

$$|P_n\rangle = \int \left(\prod_i^A d\tilde{\mathbf{p}}_i^* \right) \frac{\tilde{\Psi}_n(\{p^*\})}{\sqrt{A!}} \Phi_n(p) |\{p^*\}\rangle, \quad (62)$$

where the wave function

$$\Phi_n(p) = (2\pi)^3 \sqrt{2P_n^0} \delta^3(\mathbf{p} - \mathbf{P}) \quad (63)$$

corresponds to the nucleus with a definite three-momentum $\mathbf{P}$ and energy $P_n^0 = E_{\mathbf{p}} + \epsilon_n$, which involves the nuclear excitation energy ϵ_n . In (62), the symbol $\{p^*\}$ stands for the notation $\{p^*\} \equiv (p_1^* \dots p_n^*)$,

where p_i^* is the four-momentum of the i th nucleon in the center-of-mass system of the nucleus (at rest), and the following designation is also used:

$$d\tilde{\mathbf{p}}_i^* = \frac{d\mathbf{p}_i^*}{(2\pi)^3 \sqrt{2E_{\mathbf{p}_i^*}}}.$$

For the nuclear center-of-mass states $|n\rangle$ the normalization condition

$$\begin{aligned} \langle m|n\rangle &\equiv \int \left(\prod_i^A \frac{d\mathbf{p}_i^*}{(2\pi)^3} \right) \tilde{\Psi}_n(\{p^*\}) \\ &\times \tilde{\Psi}_m^*(\{p^*\}) (2\pi)^3 \delta^3 \left(\sum_{i=1}^A \mathbf{p}_i^* \right) = \delta_{mn}, \end{aligned} \quad (64)$$

is adopted, which for the nuclear state $|P_n\rangle$ from (62) gives a simple normalization condition

$$\langle P_m' | P_n \rangle = (2\pi)^3 2P_n^0 \delta^3(\mathbf{P} - \mathbf{P}') \delta_{nm}. \quad (65)$$

The state $|n\rangle$ corresponding to normalization (64) can be formally defined as

$$\begin{aligned} |n\rangle &= \int \left(\prod_{i=1}^A d\tilde{\mathbf{p}}_i^* \right) \frac{\tilde{\Psi}_n(\{p^*\})}{\sqrt{A!}} \\ &\times \left[(2\pi)^3 \delta^3 \left(\sum_{i=1}^A \mathbf{p}_i^* \right) \right]^{1/2} | \{p^*\} \rangle. \end{aligned} \quad (66)$$

5.2. Calculation of the Neutrino–Nucleus Scattering Amplitude

Based on the wave function of the nucleus as a bound system composed of mutually interacting nucleons (62), we can turn to calculation of the amplitude of the (anti)neutrino scattering off this system, keeping in mind that fundamental interaction events are between (anti)neutrinos and (structureless) nucleons. This assumption is acceptable, because neutrinos and antineutrinos with energies in the region of hundreds of MeV cannot “see” quarks. Therefore, it is possible to use with sufficient accuracy the effective Lagrangian of the Standard Model interaction in the four-fermion approximation

$$\mathcal{L}(x) = \frac{G_F}{\sqrt{2}} L_\mu(x) H^\mu(x), \quad (67)$$

where $L_\mu(x) = \bar{\psi}_\nu(x) \gamma_\mu (1 - \gamma_5) \psi_\nu(x)$ and

$$H^\mu(x) = \sum_{f=n,p} \bar{\psi}_f(x) \gamma^\mu (g_V^f - g_A^f \gamma_5) \psi_f(x) \quad (68)$$

are the neutrino and nucleon weak current operators written as normal products. The quantum field operators $\psi_\nu(x)$ and $\psi_{n,p}(x)$ correspond to (anti)neutrinos and nucleons.

The amplitude of the $\mathbb{S}$ matrix $\langle P_m', k' | \mathbb{S} | P_n, k \rangle$ describing the transition of the nucleus from the initial state $|P_n, k\rangle$ to the final state $\langle P_m', k'|$ as a result of interaction with the (anti)neutrino (see Fig. 2, left) can be written in the standard form (in the first order in the Fermi constant G_F)

$$\begin{aligned} \langle P_m' k' | \mathbb{S} | P_n, k \rangle &= (2\pi)^4 \delta^4(q + P_n - P_m') i \mathcal{M}_{mn}, \\ i \mathcal{M}_{mn} &= i \frac{G_F}{\sqrt{2}} l_\mu H_{mn}^\mu, \end{aligned} \quad (69)$$

where the hadronic current (matrix element) is written as

$$H_{mn}^\mu(P_n, P_m') = \langle P_m' | H^\mu(0) | P_n \rangle, \quad (70)$$

and the leptonic current, for example, for the neutrino (helicity $s = -1$), has the form

$$l_\mu(k, k') = \bar{u}(k', -1) \gamma_\mu (1 - \gamma_5) u(k, -1). \quad (71)$$

Using normalization of single-particle states (165), current (68), and antisymmetry of the nuclear wave function, one can obtain the expression for the hadronic current from (70)

$$H_{mn}^\mu(P, P') = 2\sqrt{P_n^0 P_m'^0} h_{mn}^\mu, \quad (72)$$

where the hadronic current in the “nuclear center-of-mass system” is expressed as¹¹

$$\begin{aligned} h_{mn}^\mu &= \sum_{k=1}^A \int \left(\prod_{j=1}^A \frac{d\mathbf{p}_j^*}{(2\pi)^3} \right) \frac{\bar{u}(\mathbf{p}_k + \mathbf{q}, s_k) O_k^\mu u(\mathbf{p}_k, r_k)}{\sqrt{2E_{\mathbf{p}_k} 2E_{\mathbf{p}_k + \mathbf{q}}}} \\ &\times (2\pi)^3 \delta^3 \left(\sum_{l=1}^A \mathbf{p}_l^* \right) \tilde{\Psi}_m^*(\{p_*^{(k)}\}) \tilde{\Psi}_n(\{p_*\}) \\ &\equiv \langle m | H^\mu(0) | n \rangle. \end{aligned} \quad (73)$$

Here, the designation $O_k^\mu = \gamma^\mu (g_V^k - g_A^k \gamma_5) = \gamma^\mu (g_L^k (1 - \gamma_5) + g_R^k (1 + \gamma_5))$ is introduced, and the coupling constants $g_{V,A}^k$ denote $g_{V,A}^{p/n}$ at $k = \text{proton/neutron}$. The left and right nucleon chiral coupling constants are expressed in terms of the vector and axial-vector ones

$$g_L^{p/n} = \frac{1}{2} (g_V^{p/n} + g_A^{p/n}), \quad g_R^{p/n} = \frac{1}{2} (g_V^{p/n} - g_A^{p/n}). \quad (74)$$

The arguments of the wave functions (in the nuclear rest frame) $\tilde{\Psi}_m^*(\{p_*^{(k)}\})$ and $\tilde{\Psi}_n(\{p_*\})$ are expressions defined as $\{p_*\} = (p_1^* \dots p_A^*)$, where the i th element is a pair $p_i^* = (\mathbf{p}_i^*, r_i)$, and $\{p_*^{(k)}\}$ coincides with $\{p_*\}$ except for the k th element, which is $(\mathbf{p}_k^* + \mathbf{q}, s_k)$.

¹¹ Derivation of expression (73) for the hadronic current h_{mn}^μ from (70) is given in appendix 10.2.1.

The momentum $\mathbf{p}_k$ as the argument of the Dirac spinor $u(\mathbf{p}_k)$ is the three-momentum of the k th nucleon.

The hadronic weak current of the νA scattering $h_{mn}^\mu(\mathbf{q})$ given by Eq. (73) is a sum of terms like $\bar{u}(\mathbf{p}_k + \mathbf{q}, s_k) O_k^\mu u(\mathbf{p}_k, r_k)$ corresponding to the (anti)neutrino scattering off an individual k th nucleon, which before the interaction had the three-momentum $\mathbf{p}_k$ and the projection of spin on a preferred direction r_k in the (laboratory) nuclear rest frame. The probability amplitude of finding a nucleon in the initial nuclear state $|P_n\rangle$ with these particular quantum numbers is determined by the wave function $\tilde{\Psi}_n(\{p_*\})$ given in the momentum representation and depending on the nucleon momenta in the nuclear center-of-mass system.

The scattered nucleon in the nuclear rest frame has the three-momentum $\mathbf{p}_k + \mathbf{q}$ and, generally speaking, an arbitrary projection of spin on the preferred direction s_k . The corresponding probability amplitude of finding a nucleon in the final nucleus with these particular quantum numbers is given by the expression $\tilde{\Psi}_m^*(\{p_*^{(k)}\})$.

The denominator $\sqrt{2E_{\mathbf{p}_k} 2E_{\mathbf{p}_k + \mathbf{q}}}$ depends on the energy of the initial and final nucleons in the nuclear center-of-mass system (laboratory coordinate system) and automatically takes into account the Dirac spinor normalization condition that has the form $u^\dagger(\mathbf{p}, s)u(\mathbf{p}, s) = 2E_{\mathbf{p}}$ (appendix 10.1). All three-momenta of the remaining spectator nucleons (not participating in the interaction with neutrinos) remain unchanged and may be dropped from consideration by virtue of normalization rules for the wave functions of the initial and final nuclear states (for details see appendix 10.2.1). Further transformations of the expression for the hadronic current (73) require a number of simplifying assumptions.

First, we will assume that the single-particle spinor convolution $\bar{u}(\mathbf{p}_k + \mathbf{q}, s_k) O_k^\mu u(\mathbf{p}_k, r_k)$ taken at the effective value of the nucleon three-momentum $\bar{\mathbf{p}}_k$, which is a solution of the energy and momentum conservation equation (49), can be moved outside the integral sign.

Second, we suppose that the spin and momentum dependences of the wave function $\tilde{\Psi}_n$ are not interrelated (independent) and can be written as a product of the spatial (momentum) $\tilde{\Psi}_n$ and spin χ_n terms

$$\tilde{\Psi}_n(\{p_*\}) = \tilde{\Psi}_n(\{\mathbf{p}_*\}) \chi_n(\{r\}), \quad (75)$$

which in turn depend on the variables $\{\mathbf{p}_*\} = (\mathbf{p}_1^* \dots \mathbf{p}_A^*)$ and $\{r\} = (r_1 \dots r_A)$. Spin functions are conveniently normalized by the condition

$$\chi_m^*(\{r\}) \chi_n(\{r\}) = \delta_{nm}. \quad (76)$$

As a result, expression (73) is rearranged in the form

$$h_{mn}^\mu(\mathbf{q}) = \sum_{k=1}^A \frac{\bar{u}(\bar{\mathbf{p}} + \mathbf{q}, s_k) O_k^\mu u(\bar{\mathbf{p}}, r_k)}{\sqrt{2E_{\bar{\mathbf{p}}} 2E_{\bar{\mathbf{p}} + \mathbf{q}}}} \chi_m^*(\{r^{(k)}\}) \chi_n(\{r\}) \times \int \left(\prod_{j=1}^A \frac{d\mathbf{p}_j^*}{(2\pi)^3} \right) (2\pi)^3 \delta^3 \left(\sum_{l=1}^A \mathbf{p}_l^* \right) \tilde{\Psi}_m^*(\{\mathbf{p}_*^{(k)}\}) \tilde{\Psi}_n(\{\mathbf{p}_*\}), \quad (77)$$

where $\{r^{(k)}\}$ coincides with $\{r\}$ except for the k th element, which is equal to s_k .

Third, the multidimensional integral in (77) for $h_{mn}^\mu(\mathbf{q})$ can be written in the form of a matrix element of the three-dimensional shift operator for the k th nucleon $\hat{\mathbf{X}}_k$

$$\int \left(\prod_{j=1}^A \frac{d\mathbf{p}_j^*}{(2\pi)^3} \right) (2\pi)^3 \delta^3 \left(\sum_{l=1}^A \mathbf{p}_l^* \right) \tilde{\Psi}_m^*(\{\mathbf{p}_*^{(k)}\}) \tilde{\Psi}_n(\{\mathbf{p}_*\}) = \langle m | e^{i\mathbf{q}\hat{\mathbf{X}}_k} | n \rangle \equiv f_{mn}^k(\mathbf{q}). \quad (78)$$

Derivation of (78) is given in appendix 10.2.2.

Using the definition of matrix element (69), the hadronic current (70), the form of this current in the nuclear rest frame (73), and expression (78), for matrix element (69) one has¹²

$$i\mathcal{M}_{mn} = \frac{iG_F}{\sqrt{2}} \sqrt{\frac{P_m^0 P_n^0}{E_{\bar{\mathbf{p}}} E_{\bar{\mathbf{p}} + \mathbf{q}}}} l_\mu(k, k') \sum_{k=1}^A \langle m | e^{i\mathbf{q}\hat{\mathbf{X}}_k} | n \rangle \times \chi_m^*(\{r^{(k)}\}) \chi_n(\{r\}) \bar{u}(\bar{\mathbf{p}} + \mathbf{q}, s_k) O_k^\mu u(\bar{\mathbf{p}}, r_k). \quad (79)$$

For further convenience, we designate the nuclear spin amplitude as

$$\chi_m^*(\{r^{(k)}\}) \chi_n(\{r\}) \equiv \lambda^{mn}(s, r). \quad (80)$$

In the general case, after interaction with the neutrino the spin state s of the final nucleus may differ from the spin state r of the initial nucleus. We will take the initial and final nuclear states as being eigenstates of the spin operator with the quantum numbers (J, J_3) .

¹² Actually, $i\mathcal{M}_{mn} = i \frac{G_F}{\sqrt{2}} l_\mu(k, k') H_{mn}^\mu(P_n, P_m) = i \frac{G_F}{\sqrt{2}} l_\mu(k, k') \times \sqrt{2P_n^0 2P_m^0} h_{mn}^\mu = i \frac{G_F}{\sqrt{2}} l_\mu(k, k') \sqrt{\frac{P_n^0 P_m^0}{E_{\bar{\mathbf{p}}} E_{\bar{\mathbf{p}} + \mathbf{q}}}} \sum_{k=1}^A \langle m | e^{i\mathbf{q}\hat{\mathbf{X}}_k} | n \rangle \times \chi_m^*(\{r^{(k)}\}) \chi_n(\{r\}) \bar{u}(\bar{\mathbf{p}} + \mathbf{q}, s_k) O_k^\mu u(\bar{\mathbf{p}}, r_k).$

Note that if $m = n$, then, according to (80) and normalization condition for spin wave functions (76), the amplitude $\lambda^{mn}(s, r) = \delta_{sr}$. If this amplitude is defined for the case of $m \neq n$ as λ_{sr}^{mn} , at any m, n it can be written as follows:

$$\lambda^{mn}(s, r) = \delta_{mn}\delta_{sr} + (1 - \delta_{mn})\lambda_{sr}^{mn}. \quad (81)$$

The common kinematic factor in (79) can be rearranged with separating out the leading factor m_A/m_N , where m_N and m_A are the masses of the nucleon and the nucleus, and introducing the correction $C_{mn,1}$ with the value of about unity $C_{mn,1} \sim O(1)$

$$C_{l,mn}^{1/2} = \left(\frac{P_n^0 P_m^0}{E_{\bar{p}} E_{\bar{p}+q}} \right)^{1/2} \frac{m_N}{m_A}. \quad (82)$$

Substituting (78), (80), and (82) into (79), we obtain a “final” formula of this section for the amplitude of the process $\nu A \rightarrow \nu A^{(*)}$ in the leading order in the Fermi constant

$$i\mathcal{M}_{mn} = i \frac{G_F m_A}{\sqrt{2} m_N} C_{l,mn}^{1/2} \sum_{k=1}^A \sum_{sr} f_{mn}^k \lambda^{mn}(s, r) (l, h_{sr}^k), \quad (83)$$

where

$$(l, h_{sr}^k) \equiv l_\mu(k, k') \bar{u}(\bar{\mathbf{p}} + \mathbf{q}, s_k) O_k^\mu u(\bar{\mathbf{p}}, r_k) \quad (84)$$

is the scalar product of the leptonic and k th nucleon weak neutral currents. In particular, for the neutrino one obtains

$$(l, h_{sr}^k) = \bar{u}(k', -1) \gamma_\mu (1 - \gamma_5) u(k, -1) \times \bar{u}(\bar{\mathbf{p}} + \mathbf{q}, s_k) O_k^\mu u(\bar{\mathbf{p}}, r_k).$$

Matrix element (83) is a sum of lepton–nucleon amplitudes proportional to the scalar product of the currents (l, h_{sr}^k) weighted with two factors, each being not very different from unity.

As pointed out above (definitions (75), (76), and (80)), the factor $\lambda^{mn}(s, r)$ is a spin amplitude of the transition between the nuclear states $|n\rangle$ and $|m\rangle$ and depends on the initial r and final s values of the (doubled) projection of the nuclear spin on a selected direction. The second factor given by (78) is an expression defining the nucleon form factor of the transition of the nucleus from the state n to the state m .

Note that general consideration of the matrix element $i\mathcal{M}_{mn}$ (83) corresponds to the process $\nu A \rightarrow \nu A^*$ where all possible nuclear transitions should be considered, because neither the initial $|n\rangle$ nor the final $|m\rangle$ internal quantum state of the nucleus is known in advance.

Based on the definition of $f_{mn}^k(\mathbf{q})$ in the form (78) and the symmetry properties of the nuclear wave function, one can conclude that, as mentioned above, the

form factor f_{mn}^k does not depend on the value of k but only depends on whether k corresponds to the proton or the neutron (or, in other words, on the type of nucleon indicated by k). Therefore, all amplitudes in (83) have the same phase and can thus be considered coherent using the universally adopted terminology.

Further, the expression $f_{mn}^k(\mathbf{q}) = \langle m | e^{i\mathbf{q}\hat{\mathbf{x}}_k} | n \rangle$ is a generalization of the quantum-mechanical factor $e^{i\mathbf{q}\mathbf{x}_k}$ used by Freedman [1] (Section 3). Nevertheless, there is a fundamental difference between $\langle m | e^{i\mathbf{q}\hat{\mathbf{x}}_k} | n \rangle$ and $e^{i\mathbf{q}\mathbf{x}_k}$. When $e^{i\mathbf{q}\mathbf{x}_k}$ “works” explicitly, it is assumed that the k th nucleon has a quite definite position (nailed) in the nucleus given by the coordinate $\mathbf{x}_k$. This assumption is valid for a free particle described by a plane wave. However, scattering off one free nucleon cannot result in the entire nucleus moving as a single whole because nucleons should be bound together, otherwise the nucleus will not be a single whole. The plane-wave approach does not give a correct description of the situation where nucleons should be bound together, maintaining integrity of the nucleus. In contrast, wave functions of the bound state do not rely on the presence of (do not have) fixed spatial positions of nucleons in the nucleus, and for this reason the matrix element $\langle m | e^{i\mathbf{q}\hat{\mathbf{x}}_k} | n \rangle$ does not depend on k (Section 3).

Moreover, the matrix element $\langle m | e^{i\mathbf{q}\hat{\mathbf{x}}_k} | n \rangle$ does not depend on the coordinate (of the k th nucleon), which disappears in the process of integration.

It is this quantity $f_{mn}^k(\mathbf{q}) = \langle m | e^{i\mathbf{q}\hat{\mathbf{x}}_k} | n \rangle$ given by (78) that is crucial for establishing interrelation of coherent and incoherent regimes in the elastic and inelastic (anti)neutrino–nucleus scattering and for understanding the emergence of the quadratic and linear dependences of the observed cross sections on the number of the nuclear nucleons.

Indeed, in elastic scattering there is $\lim_{q \rightarrow 0} \langle n | e^{i\mathbf{q}\hat{\mathbf{x}}_k} | n \rangle \rightarrow 1$, and, according to (83), a quadratic dependence of the cross section on the number of nucleons takes place. When $\mathbf{q} \rightarrow \infty$, this matrix element disappears, $\lim_{q \rightarrow \infty} \langle n | e^{i\mathbf{q}\hat{\mathbf{x}}_k} | n \rangle \rightarrow 0$, and with it the entire elastic scattering. Therefore, elastic scattering was named coherent in Freedman’s terms.

In the case of inelastic scattering, by virtue of the normalization condition $\langle m | n \rangle = 0$ for $n \neq m$ (64) there is $\lim_{q \rightarrow 0} \langle m | e^{i\mathbf{q}\hat{\mathbf{x}}_k} | n \rangle \rightarrow 0$ when $\mathbf{q} \rightarrow 0$. That is, inelastic scattering is zero. For $\mathbf{q} \neq 0$ the matrix element is usually $\langle m | e^{i\mathbf{q}\hat{\mathbf{x}}_k} | n \rangle \neq 0$, and, as shown in Section 6, the cross section is a linear function of the number of neutrons in the nucleus (when all possible initial and final states are considered). Since this result can be obtained by summing absolute values of squares of amplitudes, it can be called incoherent scattering.

6. CROSS SECTION OF (ANTI)NEUTRINO SCATTERING OFF NUCLEI

6.1. From the General Definition of the Cross Section to the $\nu A \rightarrow \nu A^*$ Scattering Cross Section

As is known (see for example [41, 42]), the probability of the transition from the initial state $|p\rangle$ containing s particles with four-momenta $p_i = (E_i, \mathbf{p}_i)$ to the final state $|p'\rangle$ containing r particles with four-momenta $p'_j = (E'_j, \mathbf{p}'_j)$ per unit time in unit volume is

$$dw = (2\pi)^4 \delta^4 \left(\sum_{i=1}^s p_i - \sum_{j=1}^r p'_j \right) \times \frac{n_1 \dots n_s}{2E_1 \dots 2E_s} |\mathcal{M}|^2 \prod_{j=1}^r \frac{d\mathbf{p}'_j}{(2\pi)^3 2E'_j}, \quad (85)$$

where $d\mathbf{p} = |\mathbf{p}|^2 d|\mathbf{p}| d\Omega = |\mathbf{p}|^2 d|\mathbf{p}| \sin \theta d\theta d\phi = -|\mathbf{p}|^2 d|\mathbf{p}| d\cos \theta d\phi$, θ and ϕ are the emission angles of the corresponding particle, and n_i are the densities of the beams of initial particles. Next, the process amplitude $\mathcal{M}$ is given by the relation

$$\langle \text{out} | S - 1 | \text{in} \rangle = i(2\pi)^4 \delta^4 \left(\sum p - \sum p' \right) \mathcal{M}, \quad (86)$$

where the initial $|\text{in}\rangle$ and final $\langle \text{out} |$ states are normalized in a relativistically invariant way (see appendix 10.1)

$$|\text{in}\rangle = (2\pi)^{3s/2} \sqrt{2E_1 \dots 2E_s} a^+(\mathbf{p}_1) \dots a^+(\mathbf{p}_s) |0\rangle,$$

which ensures relativistic invariance of amplitude (86).

For the scattering process $a + b \rightarrow X$ with only two particles in the initial state, it is convenient to introduce an invariant cross section $d\sigma$ defined by the expression

$$dw = d\sigma \frac{n_1 n_2}{E_1 E_2} I, \quad \text{where} \quad I = \sqrt{(p_1 p_2)^2 - m_1^2 m_2^2}.$$

In the case of the $2 \rightarrow 2$ scattering like $\nu(p_1) + A(p_2) \rightarrow \nu(p'_1) + A^{(*)}(p'_2)$, after integration over the angle $d\phi$ and over $d\mathbf{p}'_2$ using the delta function, the above formula for the differential probability dw and (85) allow obtaining an expression for the cross section¹³

$$d\sigma \equiv \frac{dw E_1 E_2}{n_1 n_2 I} = -2\pi \frac{\delta(E_1 + E_2 - E'_1 - E'_2) |\mathcal{M}|^2 |\mathbf{p}'_1|^2 d|\mathbf{p}'_1| d\cos \theta}{64\pi^2 I E'_1 E'_2},$$

from which, in view of (anti)neutrino masslessness (i.e., $|\mathbf{p}'_1| = E'_1$ and $I = E_1 p_2^0$), a formula is derived for the differential cross section of the (anti)neutrino–nucleus scattering

$$\frac{d\sigma}{dE'_1 d\cos \theta} = -\frac{\delta(E_1 + E_2 - E'_1 - E'_2) |\mathcal{M}|^2 (E'_1)^2}{32\pi (E_1 p_2^0) E'_1 E'_2} = -\frac{E'_1}{E_1} \frac{|\mathcal{M}|^2}{32\pi p_2^0 E'_2} \delta(E_1 + E_2 - E'_1 - E'_2 (E'_1)). \quad (87)$$

Considering that in the laboratory coordinate system the nucleus is supposed to be at rest and in an internal quantum state n before the interaction, for the kinematic variables in (87) we have

$$E_1 = E_\nu, \quad E'_1 = E'_\nu, \quad p_2^0 = E_A = m_A + \varepsilon_n,$$

$$E'_2 = E'_A = m_A + \varepsilon_m + T_A,$$

$$E_A - E'_A = \varepsilon_n - \varepsilon_m - T_A.$$

Then, with $\Delta\varepsilon_{mn} = \varepsilon_m - \varepsilon_n$ (see (52) from Section 4), expression (87) with matrix element (69) takes the form

$$\frac{d^2\sigma_{mn}}{dE'_\nu d\cos \theta} = -\frac{E'_\nu |i\mathcal{M}_{mn}|^2}{2^5 \pi E_\nu (m_A + \varepsilon_n)} \times \frac{\delta(E_\nu - E'_\nu - T_A(E'_\nu) - \Delta\varepsilon_{mn})}{m_A + T_A + \varepsilon_m}, \quad (88)$$

where E'_ν is given by (75), and T_A is the kinetic energy of the scattered nucleus (formulas (56)–(58)). Then, since the kinetic energy of nuclear recoil (see (56) in Section 4)

$$T_A = \sqrt{m_A^2 + \mathbf{q}^2} - m_A = \sqrt{m_A^2 + E_\nu^2 + (E'_\nu)^2 - 2E'_\nu E_\nu \cos \theta} - m_A \quad (89)$$

depends on the energy carried away by the neutrino, E'_ν , integration of the delta function in (224) should be carefully performed in the form

$$\delta(f(E'_\nu)) = \frac{\delta(E'_\nu - \tilde{E}'_\nu)}{|df(E'_\nu)/dE'_\nu|} = \frac{\delta(E'_\nu - \tilde{E}'_\nu)}{|-1 - dT_A(\tilde{E}'_\nu)/dE'_\nu|}. \quad (90)$$

Here $\tilde{E}'_\nu$ is the solution of the equation $f(E'_\nu) \equiv E_\nu - \tilde{E}'_\nu - T_A(\tilde{E}'_\nu) - \Delta\varepsilon_{mn} = 0$. For the derivative of the nuclear kinetic energy with respect to the neutrino energy one obtains

¹³The square of the matrix element $|i\mathcal{M}_{mn}|^2$ does not depend on the azimuth angle ϕ (see appendix 10.4 or formulas from Section 6.3), and therefore the integration was carried out over it.

$$\begin{aligned} \frac{dT_A(E'_v)}{dE'_v} &= \frac{E'_v - E_v \cos \theta}{\sqrt{m_A^2 + E_v^2 + (E'_v)^2 - 2E'_v E_v \cos \theta}} \\ &= \frac{E'_v - E_v \cos \theta}{T_A(E'_v) + m_A}. \end{aligned}$$

In the latter equality, the square root in the denominator is replaced with $T_A(E'_v) + m_A$ by virtue of condition (89). Since $f(\tilde{E}'_v) = 0$, one can substitute $E_v - T_A(\tilde{E}'_v) - \Delta\epsilon_{mn}$ for E'_v and obtain for $\frac{dT_A(E'_v)}{dE'_v} + 1$ from (226)

$$\begin{aligned} &\frac{dT_A(\tilde{E}'_v)}{dE'_v} + 1 \\ &= \frac{E_v - T_A(\tilde{E}'_v) - \Delta\epsilon_{mn} - E_v \cos \theta + T_A + m_A}{T_A + m_A} \\ &= \frac{m_A + E_v(1 - \cos \theta) - \Delta\epsilon_{mn}}{T_A + m_A}. \end{aligned}$$

Considering expression (57) from Section 4 for T_A in the form

$$T_A = \frac{E_v(E_v - \Delta\epsilon_{mn})(1 - \cos \theta) + \Delta^2\epsilon_{mn}/2}{m_A + E_v(1 - \cos \theta) - \Delta\epsilon_{mn}}, \quad (91)$$

one also obtains

$$\begin{aligned} E'_v &= E_v - \Delta\epsilon_{mn} - T_A(E'_v) \\ &= E_v - \Delta\epsilon_{mn} \\ &- \frac{E_v(E_v - \Delta\epsilon_{mn})(1 - \cos \theta) + \Delta^2\epsilon_{mn}/2}{m_A + E_v(1 - \cos \theta) - \Delta\epsilon_{mn}}. \end{aligned} \quad (92)$$

As a result, cross section (224) can be rearranged in the form of the integral over E'_v

$$\begin{aligned} \frac{d\sigma_{mn}}{d\cos\theta} &= \int dE'_v \frac{(-1)E'_v |i\mathcal{M}_{mn}|^2}{2^5\pi E_v(m_A + \epsilon_n)} \\ &\times \frac{\delta(E_v - E'_v - T_A(E'_v) - \Delta\epsilon_{mn})}{m_A + T_A + \epsilon_m} \\ &= \int dE'_v \frac{(-1)E'_v |i\mathcal{M}_{mn}|^2}{2^5\pi E_v(m_A + \epsilon_n)} \frac{1}{m_A + T_A + \epsilon_m} \\ &\times \frac{\delta(E'_v - \tilde{E}'_v)}{[1 + dT_A(\tilde{E}'_v)/dE'_v]} = \int dE'_v \frac{(-1)E'_v |i\mathcal{M}_{mn}|^2}{2^5\pi E_v(m_A + \epsilon_n)} \\ &\times \frac{\delta(E'_v - \tilde{E}'_v)}{m_A + T_A + \epsilon_m} \frac{T_A + m_A}{m_A + E_v(1 - \cos \theta) - \Delta\epsilon_{mn}}. \end{aligned}$$

Integration is easily performed using the $\delta(E'_v - \tilde{E}'_v)$ function, i.e., the differential cross section depending on the initial energy E_v and the neutrino scattering angle θ is

$$\begin{aligned} \frac{d\sigma_{mn}}{d\cos\theta} &= -\frac{\tilde{E}'_v |i\mathcal{M}_{mn}|^2}{2^5\pi E_v(m_A + \epsilon_n)(m_A + T_A + \epsilon_m)} \frac{1}{(m_A + T_A)} \\ &\times \frac{1}{m_A + E_v(1 - \cos \theta) - \Delta\epsilon_{mn}}. \end{aligned} \quad (93)$$

Here T_A and $\tilde{E}'_v$ are defined by expressions (91) and (92) in terms of E_v and $\cos \theta$ (and also m_A and $\Delta\epsilon_{mn}$).

Formula (93) can be simplified by ignoring $E_v(1 - \cos \theta) - \Delta\epsilon_{mn}$ as compared to m_A , i.e., by assuming $m_A \gg E_v(1 - \cos \theta) - \Delta\epsilon_{mn}$ in the formula for the nuclear kinetic energy

$$\begin{aligned} T_A &= \frac{E_v(E_v - \Delta\epsilon_{mn})(1 - \cos \theta) + \Delta^2\epsilon_{mn}/2}{m_A + E_v(1 - \cos \theta) - \Delta\epsilon_{mn}} \\ &\simeq \frac{2E_v(E_v - \Delta\epsilon_{mn})(1 - \cos \theta) + \Delta^2\epsilon_{mn}}{2m_A}. \end{aligned} \quad (94)$$

As a result, the following expression is obtained for the differential cross section (in nuclear recoil energy) with allowance for the transition from $d\cos\theta$ to dT_A in the form $E_v(E_v - \Delta\epsilon_{mn})d\cos\theta = -m_A dT_A$ (see (94)):

$$\begin{aligned} \frac{d\sigma_{mn}}{dT_A} &= \frac{d\sigma_{mn}}{d\cos\theta} \frac{d\cos\theta}{dT_A} \frac{d\cos\theta}{d\cos\theta} \frac{m_A}{E_v} \frac{-1}{E_v - \Delta\epsilon_{mn}} \\ &= \frac{|i\mathcal{M}_{mn}|^2}{2^5\pi E_v(m_A + \epsilon_n)(m_A + T_A + \epsilon_m)} \frac{\tilde{E}'_v}{(m_A + T_A)} \frac{m_A}{E_v} \frac{1}{E_v - \Delta\epsilon_{mn}}, \end{aligned}$$

which can be written in a compact form

$$\frac{d\sigma_{mn}}{dT_A} = \frac{|i\mathcal{M}_{mn}|^2}{2^5\pi E_v^2 m_A} C_{2,mn}(T_A), \quad (95)$$

by introducing the coefficient (of the order of unity)

$$\begin{aligned} C_{2,mn} &= \frac{E'_v}{E_v - \Delta\epsilon_{mn}} \frac{\left(1 + \frac{T_A}{m_A}\right) \left(1 + \frac{\epsilon_n}{m_A}\right)^{-1}}{\left(1 + \frac{T_A + \epsilon_m}{m_A}\right)} \\ &\times \left(1 + \frac{E_v(1 - \cos \theta) - \Delta\epsilon_{mn}}{m_A}\right)^{-1}. \end{aligned} \quad (96)$$

For completeness, we give the expression for differential cross section (93) without using the (simplifying) approximation for the kinetic energy of nuclear

recoil in the form (94). In this case, cross section (93) can also be reduced to the form (95)

$$\frac{d\sigma_{mn}}{dT_A} = \frac{|i\mathcal{M}_{mn}|^2}{2^5 \pi E_v^2 m_A} \tilde{C}_{2,mn}(T_A), \quad (97)$$

but with a slightly different coefficient $O(1)$ ¹⁴

$$\tilde{C}_{2,mn} = \left[1 + \frac{\epsilon_n}{m_A}\right]^{-1} \left[1 + \frac{T_A}{m_A}\right] \left[1 + \frac{T_A + \epsilon_m}{m_A}\right]^{-1}. \quad (98)$$

The result of this section is the expression for the differential cross section of the process

$$\nu(p_1) + A(p_2) \rightarrow \nu(p'_1) + A^{(*)}(p'_2),$$

defined by relation (95), where the coefficient $C_{2,mn}$ is given by Eq. (96). Or in a more general form of (97) and (98) respectively.

6.2. Coherent and Incoherent Contributions to the Cross Section for $\nu A \rightarrow \nu A^*$ Scattering

In this section, the above qualitative discussion (see Section 3) of the interrelation between the coherent and incoherent contributions to the neutrino and antineutrino scattering off the nucleus will be converted into the quantitative form, i.e., an expression simultaneously considering the coherent and incoherent contributions will be obtained for differential cross section (95)

$$\frac{d\sigma_{mn}}{dT_A} = \frac{|i\mathcal{M}_{mn}|^2}{2^5 \pi E_v^2 m_A} C_{2,mn}. \quad (99)$$

The observed differential cross sections for $\nu A \rightarrow \nu A^{(*)}$ and $\bar{\nu} A \rightarrow \bar{\nu} A^{(*)}$ can be obtained by averaging over all possible initial states of the target nucleus $|n\rangle$ and summing over all allowed final (internal) states of the recoil nucleus $|m\rangle$

$$\frac{d\sigma}{dT_A} = \sum_{n,m} \omega_n \frac{d\sigma_{mn}}{dT_A}, \quad (100)$$

where ω_n denotes the statistical weight (probability) of finding the target nucleus in the initial internal quantum state $|n\rangle$ under normal experimental conditions. Further consideration does not necessarily require explicit expressions for ω_n , and it is only important that they are normalized by the condition (of probability completeness)

$$\sum_n \omega_n = 1. \quad (101)$$

Thus, substituting expressions (83) for the matrix element and (95) for the differential cross section into

the formula for the observable differential cross section (100), we obtain

$$\begin{aligned} \frac{d\sigma}{dT_A} &= \sum_{n,m} \omega_n \frac{d\sigma_{mn}}{dT_A} = \sum_{n,m} \omega_n \frac{|i\mathcal{M}_{mn}(\mathbf{q})|^2}{2^5 \pi E_v^2 m_A} \\ &\times C_{2,mn}(\mathbf{q}) \equiv \frac{G_F^2 m_A}{2^6 \pi m_N^2 E_v^2} [T_{m=n} + T_{m \neq n}], \end{aligned} \quad (102)$$

where, for simplicity, the dependence on $\mathbf{q}$ is temporarily omitted and the following notation is introduced:

$$\begin{aligned} T_{m=n} &= \sum_{k,j}^A \sum_n \omega_n \\ &\times \left[C_{nn,1} C_{nn,2} f_{nn}^k f_{nn}^{j*} \sum_r (l, h_{rr}^k) \sum_s (l, h_{ss}^j)^\dagger \right], \\ T_{m \neq n} &= \sum_{k,j}^A \sum_n \omega_n \left[\sum_{m \neq n} C_{mn,1} C_{mn,2} f_{mn}^k f_{mn}^{j*} \right. \\ &\times \left. \sum_{sr} \lambda_{sr}^{mn}(l, h_{sr}^k) \left(\sum_{s'r'} \lambda_{s'r'}^{mn}(l, h_{s'r'}^j) \right)^\dagger \right]. \end{aligned} \quad (103)$$

Here the separate term $T_{m=n}$ denotes the contribution corresponding to the unchanged initial state of the nucleus, i.e., $n = m$, in which case the projection of the spin of the interacting (active) nucleon does not change, because at $m = n$ spin amplitudes yield the delta function $\lambda_{s,r}^{nn} = \delta_{s,r}$ (formula (81)) and may be dropped from further consideration.

The key role in (102) is played by the scalar products of the four-vector weak leptonic and nucleon currents $(l, h_{sr}^{p/n})$, where the leptonic current components $l^\mu(k, k')$, e.g., for neutrinos, are given by Eq. (71), and the nucleon current has the form

$$(h_{sr}^{p/n})_\mu \bar{u}(\bar{\mathbf{p}} + \mathbf{q}, s) O_\mu^{p/n} u(\bar{\mathbf{p}}, r), \quad (105)$$

where $\bar{\mathbf{p}}$ is the solution of the equation for the nuclear energy–momentum conservation (49)

$$\begin{aligned} &\sqrt{m_N^2 + \bar{\mathbf{p}}^2 + 2\bar{\mathbf{p}}\mathbf{q} + \mathbf{q}^2} \\ &- \sqrt{m_N^2 + \bar{\mathbf{p}}^2} = \frac{\mathbf{q}^2}{2m_A} + \Delta\epsilon_{mn}. \end{aligned}$$

Looking slightly ahead, we note that each of the squares of the scalar products $|(l, h_{sr}^{p/n})|^2$ from section 10.4 (see also section 6.3) involves a common term $64(s - m_N^2)^2$, where $s = (p + k)^2$ is the square of the total energy of the neutrino and the nucleon in their common center of mass, and m_N is the nucleon mass (see appendix 10.4). In the leading (nonrelativistic) approximation this term can be approximated by the formula $2^8 m_N^2 E_v^2$. To achieve exact correspondence

¹⁴For details see section 10.3.

between these expressions, we introduce the correction factor $C_{3,mn}$, which reflects the fact that the nucleon in the initial state has a nonzero three-momentum (according to (49)), i.e., we assume that

$$(s - m_N^2)^2 = 4m_N^2 E_\nu^2 C_{3,mn}. \quad (106)$$

In what follows, all correction factors will be denoted by a single symbol g^{mn} ,

$$g^{mn} = C_{1,mn} C_{2,mn} C_{3,mn}. \quad (107)$$

This factor is not very different from unity either, but it has different values for the first (coherent) and second (incoherent) terms in (102). Let us take these terms out of the double summation sign in (102), considering that the effective value of this correction factor is $g_{\text{coh}} = g_c$ for the coherent term and $g_{\text{inc}} = g_i$ for the incoherent term.

It is worth mentioning that summation over the index n in the first term of (102), i.e., in (103), gives rise to form factors averaged over all possible initial states of the nucleus

$$\sum_n \omega_n f_{nn}^k f_{nn}^{j*} = \begin{cases} |F_{p/n}(\mathbf{q})|^2, & \text{when } (k, j) \text{ are equal to either } pp \text{ or } nn; \\ F_p(\mathbf{q}) F_n^*(\mathbf{q}), & \text{when } (k, j) \text{ are equal to } pn; \\ F_n(\mathbf{q}) F_p^*(\mathbf{q}), & \text{when } (k, j) \text{ are equal to } np. \end{cases} \quad (108)$$

In other words, the left-hand side of this relation is equal to the square of the modulus of the nuclear proton or neutron form factor $|F_{p/n}(\mathbf{q})|^2$ if the summation indices k and j correspond either to a pair of protons or to a pair of neutrons. If these indices do not simultaneously correspond to a pair of identical nucleons, there arises a product of the nuclear proton and neutron form factors in the form $F_p(\mathbf{q}) F_n^*(\mathbf{q})$ or

$F_n(\mathbf{q}) F_p^*(\mathbf{q})$. That is why expression (103) is proportional to the square of the modulus of the proton and neutron contributions

$$T_{m=n} \propto \left| \sum_{k=1}^Z \sum_r (l, h_{rr}^p) F_p + \sum_{k=1}^N \sum_r (l, h_{rr}^n) F_n \right|^2. \quad (109)$$

In more detail, this term is rearranged as

$$\begin{aligned} T_{m=n}(\mathbf{q}) &= \left| \sum_k \sqrt{g_{\text{coh}}^p} \sum_r (l, h_{rr}^p(\mathbf{q})) F_p(\mathbf{q}) + \sum_k \sqrt{g_{\text{coh}}^n} \sum_r (l, h_{rr}^n(\mathbf{q})) F_n(\mathbf{q}) \right|^2 \\ &= \left| \sum_{f=p,n} \sqrt{g_{\text{coh}}^f} \sum_{k=1}^{A_f} \sum_r (l, h_{rr}^f(\mathbf{q})) F_f(\mathbf{q}) \right|^2. \end{aligned} \quad (110)$$

Here are introduced correction coefficients $g_{\text{coh}}^{p/n} = C_{nn,1}^{p/n} C_{nn,2}^{p/n}$ for the coherent term in (102) and A_f that denotes the total number of nucleons, $f = p, n$, in the nucleus.

We now turn to consideration of the second (expression (104)) (incoherent) term $T_{m \neq n}$ in the formula for the observed cross section $\frac{d\sigma}{dT_A}$, which involves summation over the indices m, n . This summation cannot be performed without model assumptions about the behavior of the spin amplitudes λ_{sr}^{mn} .

If we assume, for example, that the amplitudes λ_{sr}^{mn} do not depend on the indices m, n at all, the summation over m, n can be performed as follows.

Let us begin with a simple case where k and j “indicate” the same type of nucleon, say, the proton. If $k = j$, then, in view of the nuclear state completeness condition $\sum_m |m\rangle \langle m| = \hat{I}$, probability conservation condition (101) $\sum_n \omega_n = 1$, nuclear state normalization condition (64) $\langle m|n\rangle = \delta_{mn}$, definition (78) for $f_{mn}^k(\mathbf{q}) = \langle m|e^{i\mathbf{q}\mathbf{X}_k}|n\rangle$, and relations for their squares (108), it is possible to carry out all summations, obtaining

$$\begin{aligned} \sum_n \omega_n \sum_{m \neq n} f_{mn}^k f_{mn}^{k*} &= \sum_n \omega_n \left[\sum_m f_{mn}^k f_{mn}^{k*} - f_{nn}^k f_{nn}^{k*} \right] \\ &= \sum_n \omega_n \left[\langle n|e^{i\mathbf{q}\mathbf{X}_k} \sum_m |m\rangle \langle m| e^{-i\mathbf{q}\mathbf{X}_k} |n\rangle \right] - |F_p(\mathbf{q})|^2 \\ &= 1 - |F_p(\mathbf{q})|^2. \end{aligned} \quad (111)$$

If $k \neq j$, then, considering the same conditions as in derivation of (111), for protons (index p) there is

$$\sum_n \omega_n \sum_{m \neq n} f_{mn}^k f_{mn}^{j*} = \langle \text{cov}(e^{iq\hat{X}_k}, e^{-iq\hat{X}_j}) \rangle_p. \quad (112)$$

On the right-hand side of (112) the covariance of quantum shift operators $e^{-iq\hat{X}_j}$ and $e^{iq\hat{X}_k}$ for the $|n\rangle$ state is introduced in the form

$$\begin{aligned} \text{cov}_{nn}(e^{iq\hat{X}_k}, e^{-iq\hat{X}_j}) &= \langle n | e^{iq\hat{X}_k} e^{-iq\hat{X}_j} | n \rangle \\ &- \langle n | e^{iq\hat{X}_k} | n \rangle \langle n | e^{-iq\hat{X}_j} | n \rangle. \end{aligned} \quad (113)$$

Here and in (111) the fact was used that the “incomplete” sum over m , $\sum_{m \neq n} f_{mn}^k f_{mn}^{j*}$, can be “completed” and written as $\sum_m f_{mn}^k f_{mn}^{j*} - \sum_{m=n} f_{mn}^k f_{mn}^{j*}$, and using the completeness condition $\sum_m |m\rangle \langle m| = \hat{I}$, one can obtain the expression (113)

$$\begin{aligned} \sum_{m \neq n} f_{mn}^k f_{mn}^{j*} &= \sum_m \langle n | e^{iq\hat{X}_k} | m \rangle \langle m | e^{-iq\hat{X}_j} | n \rangle \\ &- \sum_{m=n} \langle n | e^{iq\hat{X}_k} | m \rangle \langle m | e^{-iq\hat{X}_j} | n \rangle \\ &= \langle n | e^{iq\hat{X}_k} e^{-iq\hat{X}_j} | n \rangle - \langle n | e^{iq\hat{X}_k} | n \rangle \langle n | e^{-iq\hat{X}_j} | n \rangle. \end{aligned}$$

The averaging symbol $\langle \dots \rangle$ in (112) is defined as

$$\langle \text{cov}(e^{iq\hat{X}_k}, e^{-iq\hat{X}_j}) \rangle_p = \sum_n \omega_n \text{cov}_{nn}(e^{iq\hat{X}_k}, e^{-iq\hat{X}_j}). \quad (114)$$

Expression (114) vanishes both at small transferred momenta $\mathbf{q} \rightarrow 0$ and at large ones $\mathbf{q} \rightarrow \infty$

$$\begin{aligned} \lim_{\mathbf{q} \rightarrow 0} \langle \text{cov}(e^{-iq\hat{X}_j}, e^{iq\hat{X}_k}) \rangle_p &= 0, \\ \lim_{\mathbf{q} \rightarrow \infty} \langle \text{cov}(e^{-iq\hat{X}_j}, e^{iq\hat{X}_k}) \rangle_p &= 0. \end{aligned} \quad (115)$$

It is believed that covariance functions like (112) can be ignored. For example, in nuclear shell models where multiparticle wave functions of nuclei are constructed in the form of products of single-particle wave functions, covariance (112) becomes identically zero. Counting in favor of smallness of covariance (112) is the fact that the inelastic scattering cross section can be approximated with a good accuracy by the linear dependence on the number of the target nucleons.

A similar consideration easily applies to neutrons as well. It is not a problem to generalize this consideration to the joint case of protons and neutrons. Let, for example, k correspond to the proton and j to the neutron, and $k \neq j$. Then,

$$\sum_n \omega_n \sum_{m \neq n} f_{mn}^k f_{mn}^{j*} = \langle \text{cov}(e^{iq\hat{X}_k}, e^{-iq\hat{X}_j}) \rangle_{pn}. \quad (116)$$

It is clear from the previous discussion that this contribution to the observed cross section $\frac{d\sigma}{dT_A}$ (formula (102)) is also negligibly small.

In further consideration covariance contributions like (112) and (116) are neglected. In other words, all these covariance contributions are taken to be zero.

As was already mentioned, further transformation of the (incoherent) term $T_{m \neq n}$ (104) requires certain assumptions about the unknown behavior of spin amplitudes λ_{sr}^{mn} . These amplitudes will be assumed equal to their respective mean values for protons and neutrons and no longer dependent on the internal nuclear state

$$\lambda_{sr}^{mn} \approx \lambda_{sr}^{p/n}. \quad (117)$$

In view of this simplifying assumption and neglected contributions like (112) and (116), incoherent term (104) can be rearranged as

$$\begin{aligned} T_{m \neq n}(\mathbf{q}) &\equiv \sum_{k,j} \sum_n \omega_n C_{mn,1} C_{mn,2} \sum_{m \neq n} f_{mn}^k f_{mn}^{j*} \\ &\times \sum_{s,r} \lambda_{sr}^{mn}(l, h_{sr}^k) \left(\sum_{s',r'} \lambda_{s',r'}^{mn}(l, h_{s',r'}^j) \right)^\dagger \\ &= g_{\text{inc}}^p \sum_{k=1}^Z \sum_{s,r} |\lambda_{sr}^p|^2 |(l, h_{sr}^p)|^2 [1 - |F_p|^2] \\ &+ g_{\text{inc}}^n \sum_{k=1}^N \sum_{s,r} |\lambda_{sr}^n|^2 |(l, h_{sr}^n)|^2 [1 - |F_n|^2] \\ &\equiv \sum_{f=p,n} g_{\text{inc}}^f \sum_{k=1}^{A_f} \sum_{s,r} |\lambda_{sr}^f|^2 |(l, h_{sr}^f(\mathbf{q}))|^2 [1 - |F_f(\mathbf{q})|^2]. \end{aligned} \quad (118)$$

Here “incoherent” correction factors $g_{\text{inc}}^{p/n} = C_{mn,1}^{p/n} C_{mn,2}^{p/n}$ are introduced.

Thus, with approximation (106) considered, the approximation for differential cross section (102) can be written in the form of two fundamentally different terms

$$\begin{aligned} \frac{d\sigma}{dT_A} &= \frac{4G_F^2 m_A}{\pi} \left\{ g_c \left| \sum_{f=n,p} \sum_{k=1}^{A_f} \sum_r (l, h_{sr}^f) F_f \right|^2 \right. \\ &+ g_i \sum_{f=n,p} \sum_{k=1}^{A_f} \sum_{s,r} |\lambda_{sr}^f|^2 |(l, h_{sr}^f)|^2 (1 - |F_f|^2) \left. \right\}, \end{aligned} \quad (119)$$

where $|\lambda_{sr}^{p/n}|^2$ and $g_{i/c}$ are defined by factoring out of the double summation sign $\sum_{nm}$ in (102) and (100), respectively, in accordance with (81) and (117) for $|\lambda_{sr}^{mn}|^2$ and (107) for g^{mn} . As was already mentioned, kinematic correction functions $g_{i/c}$ are not very different from unity. Squares of nuclear form factors $|F_{p/n}|^2$ for protons and neutrons are defined by relation (78).

Summing up this section, we note the main result is expression (119) for the observed differential cross section of the (anti)neutrino–nucleus scattering. Recall that formula (119) was obtained with an

important simplifying assumption that all correlation contributions like (112) and (116) can be ignored. It should be stressed that the first and second terms in (102) and (119) correspond to the elastic and inelastic (anti)neutrino–nucleus scattering, respectively. The first term (when $\sum_{n=m}$) involves both indices k and j , which leads to the quadratic dependence of the cross section on the number of nucleons in the nucleus. The second term (when $\sum_{n \neq m}$) is proportional, in good approximation, to δ_{kj} , which automatically leads to the linear dependence of the cross section on the number of nucleons in the nucleus. In Freedman's language, the first term corresponds to coherent scattering, and the second term, to incoherent.

Moreover, summation at a level of amplitudes of scattering off individual nucleons (which is traditionally related to the coherence concept) is clearly discernible in the first term of (119). Here, each nucleon (proton and/or neutron) enters into the sum weighted with its corresponding form factor $F_{p/n}(\mathbf{q})$. It is also important that this term corresponds to the case where the nucleus does not change its spin state (projection of spin on a given direction), which is evident from the character of the summation over projections of spins of nucleons participating in the interaction, $\sum_r (l, h_{rr}^{p/n})$. All active nucleons maintain orientation of the spin after the interaction ($r' \equiv r$).

The second term in (119), traditionally associated with the incoherent contribution to the cross section, depends on the spin amplitudes $|\lambda_{sr}^{p/n}|^2$, which correspond to the probabilities (averaged over n and summed over m) for the nucleon to change its spin index from r to s in processes like $|n\rangle \rightarrow |m\rangle$. Though calculation of $|\lambda_{sr}^{p/n}|^2$ requires a nuclear model of wave functions, in this work¹⁵, a simple approximation is used which implies that they all are equal to unity, i.e., $|\lambda_{sr}^{p/n}|^2 \rightarrow 1$, or that for any initial orientation of the spin of the active nucleon (index r) any final orientation of the spin of this nucleon (index s) is equally probable.

To fulfill the last stage of derivation of the formula $\frac{d\sigma}{dT_A}$ for the observed differential cross section, it is necessary, according to (119), to have expressions for the scalar products $(l, h_{sr}^{p/n})$ of the weak leptonic and nucleon currents.

These quantities for both the ν –nucleon and $\bar{\nu}$ –nucleon elastic scattering are obtained in appendix 10.4 within the chirality basis and σ_3 basis. The latter

basis corresponds to the situation in which the projection of the nucleon spin is considered with respect to the fixed axis coinciding with the direction of the incident (anti)neutrino momentum. Though, as shown in (307), the ultimate result does not depend on the choice of the basis, it is still convenient to use the chirality basis (expressions (277) and (281)) for calculating the incoherent contribution to the cross section and the σ_3 basis (expressions (299) and (300)) for calculating the coherent contribution.

6.3. Scalar Products of Currents and Relations between Them

The procedure of calculating scalar products of currents for the neutrino–nucleon (l_ν, h_{rr}^f) and anti-neutrino–nucleon $(l_{\bar{\nu}}, h_{rr}^f)$ elastic scattering is described at length in appendix 10.4. All expressions for these quantities and relations between them, which are needed for further consideration, are given below. In the σ_3 basis (denoted by the index η) there are the following expressions for all necessary scalar products (the first index corresponds to the spin of the final nucleon and the second to the spin of the initial nucleon):

$$\begin{aligned}
 (l_\nu, h_{--}^\eta) &= 8(s - m^2) \cos \frac{\theta}{2} \mathbf{g}_R \left[1 - \left(1 - \frac{m}{\sqrt{s}} \right) \sin^2 \frac{\theta}{2} \right], \\
 (l_{\bar{\nu}}, h_{++}^\eta) &= 8(s - m^2) \cos \frac{\theta}{2} \mathbf{g}_L \left[1 - \left(1 - \frac{m}{\sqrt{s}} \right) \sin^2 \frac{\theta}{2} \right], \\
 (l_\nu, h_{+-}^\eta) &= [+e^{+i\phi}] 8(s - m^2) \\
 &\quad \times \sin \frac{\theta}{2} \left(\mathbf{g}_L - \mathbf{g}_R \frac{m}{\sqrt{s}} \left[1 - \left(1 - \frac{m}{\sqrt{s}} \right) \sin^2 \frac{\theta}{2} \right] \right), \\
 (l_{\bar{\nu}}, h_{-+}^\eta) &= [-e^{-i\phi}] 8(s - m^2) \\
 &\quad \times \sin \frac{\theta}{2} \left(\mathbf{g}_R - \mathbf{g}_L \frac{m}{\sqrt{s}} \left[1 - \left(1 - \frac{m}{\sqrt{s}} \right) \sin^2 \frac{\theta}{2} \right] \right), \\
 (l_\nu, h_{+-}^\eta) &= [-e^{-i\phi}] 8(s - m^2) \sin \frac{\theta}{2} \cos^2 \frac{\theta}{2} \mathbf{g}_R \left(1 - \frac{m}{\sqrt{s}} \right), \\
 (l_{\bar{\nu}}, h_{-+}^\eta) &= [+e^{+i\phi}] 8(s - m^2) \sin \frac{\theta}{2} \cos^2 \frac{\theta}{2} \mathbf{g}_L \left(1 - \frac{m}{\sqrt{s}} \right), \\
 (l_\nu, h_{++}^\eta) &= 8(s - m^2) \\
 &\quad \times \cos \frac{\theta}{2} \left[\mathbf{g}_L + \mathbf{g}_R \frac{m}{\sqrt{s}} \left(1 - \frac{m}{\sqrt{s}} \right) \sin^2 \frac{\theta}{2} \right], \\
 (l_{\bar{\nu}}, h_{--}^\eta) &= 8(s - m^2) \\
 &\quad \times \cos \frac{\theta}{2} \left[\mathbf{g}_R + \mathbf{g}_L \frac{m}{\sqrt{s}} \left(1 - \frac{m}{\sqrt{s}} \right) \sin^2 \frac{\theta}{2} \right].
 \end{aligned}$$

For clarity, here and elsewhere the differences in similar formulas corresponding to neutrinos and anti-neutrinos are given in bold type.

According to the approximation with the notation $a = \sin^2 \frac{\theta}{2}$ and $b = \frac{m}{\sqrt{s}}$ adopted in (106), squares of

¹⁵This approximation can be a subject of special consideration in the future.

matrix elements $|(l_{\bar{\nu}/\nu}, h_{s,s}^{\eta,f})|^2$ divided by the factor $[8^2(s-m^2)^2]$ are expressed as

$$\begin{aligned}
 |(l_{\bar{\nu}} \cdot h_{-}^{\eta})|^2 &= \cos^2 \frac{\theta}{2} g_R^2 \left[1 - \left(1 - \frac{m}{\sqrt{s}} \right) \sin^2 \frac{\theta}{2} \right]^2 \\
 &= (1-a) \mathbf{g}_R^2 [1 - (1-b)a]^2, \\
 |(l_{\bar{\nu}} \cdot h_{++}^{\eta})|^2 &= \cos^2 \frac{\theta}{2} g_L^2 \left[1 - \left(1 - \frac{m}{\sqrt{s}} \right) \sin^2 \frac{\theta}{2} \right]^2 \\
 &= (1-a) \mathbf{g}_L^2 [1 - (1-b)a]^2, \\
 |(l_{\bar{\nu}} \cdot h_{+-}^{\eta})|^2 &= \sin^2 \frac{\theta}{2} \left(g_L - g_R \frac{m}{\sqrt{s}} \left[1 - \left(1 - \frac{m}{\sqrt{s}} \right) \sin^2 \frac{\theta}{2} \right] \right)^2 \\
 &= a(\mathbf{g}_L - \mathbf{g}_R b [1 - (1-b)a])^2, \\
 |(l_{\bar{\nu}} \cdot h_{-+}^{\eta})|^2 &= \sin^2 \frac{\theta}{2} \left(g_R - g_L \frac{m}{\sqrt{s}} \left[1 - \left(1 - \frac{m}{\sqrt{s}} \right) \sin^2 \frac{\theta}{2} \right] \right)^2 \\
 &= a(\mathbf{g}_R - \mathbf{g}_L b [1 - (1-b)a])^2, \\
 |(l_{\bar{\nu}} \cdot h_{++}^{\eta})|^2 &= \sin^2 \frac{\theta}{2} \cos^4 \frac{\theta}{2} g_R^2 \left(1 - \frac{m}{\sqrt{s}} \right)^2 \\
 &= a(1-a)^2 \mathbf{g}_R^2 (1-b)^2, \\
 |(l_{\bar{\nu}} \cdot h_{+-}^{\eta})|^2 &= \sin^2 \frac{\theta}{2} \cos^4 \frac{\theta}{2} g_L^2 \left(1 - \frac{m}{\sqrt{s}} \right)^2 \\
 &= a(1-a)^2 \mathbf{g}_L^2 (1-b)^2, \\
 |(l_{\bar{\nu}} \cdot h_{++}^{\eta})|^2 &= \cos^2 \frac{\theta}{2} \left[g_L + g_R \frac{m}{\sqrt{s}} \left(1 - \frac{m}{\sqrt{s}} \right) \sin^2 \frac{\theta}{2} \right]^2 \\
 &= (1-a) [\mathbf{g}_L + \mathbf{g}_R b(1-b)a]^2, \\
 |(l_{\bar{\nu}} \cdot h_{-+}^{\eta})|^2 &= \cos^2 \frac{\theta}{2} \left[g_R + g_L \frac{m}{\sqrt{s}} \left(1 - \frac{m}{\sqrt{s}} \right) \sin^2 \frac{\theta}{2} \right]^2 \\
 &= (1-a) [\mathbf{g}_R + \mathbf{g}_L b(1-b)a]^2.
 \end{aligned}$$

As shown in appendix 10.4 (formulas (307)), there is a relation between the squares of these scalars in different bases

$$\begin{aligned}
 |(l_{\bar{\nu}}, h_{++}^{\eta})|^2 + |(l_{\bar{\nu}}, h_{+-}^{\eta})|^2 &= |(l_{\bar{\nu}}, h_{+-}^{\chi})|^2 + |(l_{\bar{\nu}}, h_{-+}^{\chi})|^2, \\
 |(l_{\bar{\nu}}, h_{+-}^{\eta})|^2 + |(l_{\bar{\nu}}, h_{-+}^{\eta})|^2 &= |(l_{\bar{\nu}}, h_{++}^{\chi})|^2 + |(l_{\bar{\nu}}, h_{--}^{\chi})|^2.
 \end{aligned} \quad (120)$$

In going from the neutrino to the antineutrino, $l_{\nu} \rightarrow l_{\bar{\nu}}$, the upper line in (120) passes into the lower line. Generalization of relations (120) to the case of total summation over all possible projections of the initial and final nucleon spins (see (310)) yields the equality

$$\sum_{r', r=\pm} |(l_{\bar{\nu}/\nu}, h_{r', r}^{\chi, f})|^2 = \sum_{s', s=\pm} |(l_{\bar{\nu}/\nu}, h_{s', s}^{\eta, f})|^2, \quad (121)$$

which allows only the above squares of scalar products obtained in the σ_3 basis to be further used.

6.4. Cross Sections for the $\nu A \rightarrow \nu A^*$ and $\bar{\nu} A \rightarrow \bar{\nu} A^*$ Scatterings

As follows from (119) given below for the sake of convenience

$$\begin{aligned}
 \frac{d\sigma}{dT_A} &= \frac{4G_F^2 m_A}{\pi} \left\{ g_c \left| \sum_{f=n,p} \sum_{k=1}^{A_f} \sum_r (l, h_{rr}^f) F_f \right|^2 \right. \\
 &\quad \left. + g_i \sum_{f=n,p} \sum_{k=1}^{A_f} \sum_{s,r} |\lambda_{sr}^f|^2 |(l, h_{sr}^f)|^2 (1 - |F_f|^2) \right\},
 \end{aligned}$$

the observed cross sections for neutrino and antineutrino scattering off the nucleus can be written as a sum of the coherent and incoherent terms in the unified form for neutrinos and antineutrinos

$$\frac{d\sigma^{\bar{\nu}, \nu}}{dT_A} = \frac{d\sigma_{\text{inc}}^{\bar{\nu}, \nu}}{dT_A} + \frac{d\sigma_{\text{coh}}^{\bar{\nu}, \nu}}{dT_A} \equiv \frac{4G_F^2 m_A}{\pi} \left(\frac{d\sigma_{\text{inc}}^{\bar{\nu}, \nu}}{dT_A} + \frac{d\sigma_{\text{coh}}^{\bar{\nu}, \nu}}{dT_A} \right), \quad (122)$$

where the incoherent term is expressed (provided that $|\lambda_{sr}^f|^2 = 1$) as

$$\begin{aligned}
 \frac{d\sigma_{\text{inc}}^{\bar{\nu}, \nu}}{dT_A} &= \sum_{f=p,n} g_{\text{inc}}^f (1 - |F_f|^2) \sum_{k=1}^{A_f} \sum_{r', r} |(l_{\bar{\nu}, \nu}, h_{r', r}^{\eta, f})|^2 \\
 &= \sum_{f=p,n} g_{\text{inc}}^f (1 - |F_f|^2) \\
 &\quad \times \sum_{k=1}^{A_f} \left[\sum_{r'=\pm}^{\text{out}} |(l_{\bar{\nu}, \nu}, h_{r', -}^{\eta, f})|^2 + \sum_{r'=\pm}^{\text{out}} |(l_{\bar{\nu}, \nu}, h_{r', +}^{\eta, f})|^2 \right].
 \end{aligned} \quad (123)$$

According to (121), only squares of scalar products $|(l_{\bar{\nu}, \nu}, h_{r', r}^{\eta, f})|^2$ calculated in the σ_3 basis are used in (123). The coherent cross section from (122), where the projection of the scattered nucleon spin on the preferred direction remains unchanged, has the form

$$\begin{aligned}
 \frac{d\sigma_{\text{coh}}^{\bar{\nu}, \nu}}{dT_A} &= \sum_{f=p,n} \left| \sqrt{g_{\text{coh}}^f} F_f \sum_{k=1}^{A_f} \sum_r (l_{\bar{\nu}, \nu}, h_{rr}^{\eta, f}(\mathbf{q})) \right|^2 \\
 &= \left| \sum_{f=p,n} \sqrt{g_{\text{coh}}^f} F_f \sum_{k=1}^{A_f} [(l_{\bar{\nu}, \nu}, h_{++}^{\eta, f}) + (l_{\bar{\nu}, \nu}, h_{--}^{\eta, f})] \right|^2.
 \end{aligned} \quad (124)$$

Further rearrangement of the expressions for the coherent (124) and incoherent (123) contributions to the total (anti)neutrino–nucleus scattering cross section (122) involves consideration of expressions for the scalar products of weak lepton–nucleon currents $(l_{\bar{\nu}, \nu}, h_{r', r}^{\eta, f})$ entering into them.

This procedure is detailed in appendix 10.5, whence follows the complete set of formulas for the cross sections of the neutrino scattering $\nu(k) + A \rightarrow \nu(k') + A^{(*)}$ and the antineutrino scattering $\bar{\nu}(k) + A \rightarrow \bar{\nu}(k') + A^{(*)}$ off the nuclear target [19, 20], which are sums of coherent and incoherent terms depending on the recoil energy T_A of a nucleus

$$\frac{d\sigma^{\nu/\bar{\nu}}}{dT_A} = \frac{d\sigma_{\text{inc}}^{\nu/\bar{\nu}}}{dT_A} + \frac{d\sigma_{\text{coh}}^{\nu/\bar{\nu}}}{dT_A}. \quad (125)$$

In what follows the left and upper indices indicate neutrinos, and the right and lower indices indicate

antineutrinos. In terms of $A_{\pm}^f$, i.e., of the number of the f -type nucleons with the spin projection $\pm 1/2$ on the direction of the incident lepton momentum, the coherent and incoherent terms from (125) are

$$\begin{aligned} \frac{d\sigma_{\text{inc}}^{\nu/\bar{\nu}}}{dT_A} = & \frac{4G_F^2 m_A}{\pi} \sum_{f=p,n} g_{\text{inc}}^f (1 - |F_f(q^2)|^2) \left[\mathbf{A}_{\pm}^f (g_{R/L}^f)^2 \frac{s(1-y)^2 - m^2(1-y)}{s-m^2} \right. \\ & \left. + \left\{ \mathbf{A}_{\pm}^f \left(g_{L/R}^f - g_{R/L}^f \frac{ym^2}{s-m^2} \right)^2 + (g_{R/L}^f)^2 \frac{ym^2[s(1-y) - m^2]}{(s-m^2)^2} \right\} \right], \end{aligned} \quad (126)$$

$$\frac{d\sigma_{\text{coh}}^{\nu/\bar{\nu}}}{dT_A} = \frac{4G_F^2 m_A}{\pi} \left(1 - \frac{T_A}{T_A^{\text{max}}} \right) \left| \sum_{f=p,n} \sqrt{g_{\text{coh}}^f F_f(q^2)} \left\{ \mathbf{A}_{\pm}^f \left[g_{L/R}^f + g_{R/L}^f \frac{my}{\sqrt{s+m}} \right] + \mathbf{A}_{\mp}^f g_{R/L}^f \left[1 - \frac{\sqrt{s}y}{\sqrt{s+m}} \right] \right\} \right|^2. \quad (127)$$

In terms of $A^f = A_+^f + A_-^f$ and $\Delta A^f = A_+^f - A_-^f$, these terms are respectively¹⁶

$$\begin{aligned} \frac{d\sigma_{\text{inc}}^{\nu/\bar{\nu}}}{dT_A} = & \frac{2G_F^2 m_A}{\pi} \sum_{f=p,n} g_{\text{inc}}^f (1 - |F_f(q^2)|^2) \left\{ \mathbf{A}^f \left[g_{L/R}^f)^2 + (g_{R/L}^f)^2 (1-y)^2 - g_L^f g_R^f \frac{2m^2 y}{s-m^2} \right] \right. \\ & \left. + (\pm \Delta \mathbf{A}^f) [g_{L/R}^f - g_{R/L}^f (1-y)] \times \left[g_{L/R}^f + g_{R/L}^f (1-y) \frac{s+m^2}{s-m^2} \right] \right\}, \end{aligned} \quad (128)$$

$$\begin{aligned} \frac{d\sigma_{\text{coh}}^{\nu/\bar{\nu}}}{dT_A} = & \frac{G_F^2 m_A}{\pi} \left(1 - \frac{T_A}{T_A^{\text{max}}} \text{Big} \right) \\ & \times |G_V^{\nu/\bar{\nu}}(q^2) + G_A^{\nu/\bar{\nu}}(q^2)|^2, \quad \text{where} \\ & G_V^{\nu/\bar{\nu}}(q^2) \\ = & \sum_{f=p,n} \sqrt{g_{\text{coh}}^f F_f(q^2)} g_V^f \left[\mathbf{A}^f \left(1 - \frac{y}{2} \right) + \frac{y}{2} (\pm \Delta \mathbf{A}^f) \right] \\ & G_A^{\nu/\bar{\nu}}(q^2) = \\ & \sum_{f=p,n} \sqrt{g_{\text{coh}}^f F_f(q^2)} (\pm g_A^f) \left[\mathbf{A}^f \frac{y}{2} + \left(1 - \frac{y}{2} \right) (\pm \Delta \mathbf{A}^f) \right]. \end{aligned} \quad (129)$$

Recall that $g_{L/R}^f = \frac{1}{2}(g_V^f \pm g_A^f)$ or $g_V^f = g_L^f + g_R^f$, $g_A^f = g_L^f - g_R^f$ and the kinematic variables are given by the expressions

$$\begin{aligned} q^2 = (k - k')^2, \quad \text{and} \quad y = \frac{(p, q)}{(p, k)} = \frac{s - m^2}{s} \frac{q^2}{q_{\text{max}}^2} \\ \simeq \frac{s - m^2}{s} \frac{T_A}{T_A^{\text{max}}}, \quad E_V = \frac{s - m^2}{2\sqrt{s}}, \quad \text{and} \quad \tau = \frac{\sqrt{s} - m}{\sqrt{s} + m}. \end{aligned}$$

Here $s = (p + k)^2$ is the total energy squared of the neutrino and nucleon, where k is the four-momentum of the initial lepton, and p is the effective four-momentum of the scattering nucleon that is a solution of nucleus energy-momentum conservation equation (49). The variables y and s are completely determined by the kinematics of the neutrino-nucleon system.

Further, $A_p = Z$, $A_n = N$, $\Delta A_p \equiv \Delta Z = Z_+ - Z_-$, and $\Delta A_n \equiv \Delta N = N_+ - N_-$, where $Z_{\pm}$ and $N_{\pm}$ are the numbers of protons and neutrons with the spin projection $\pm 1/2$ on the initial neutrino direction. The correction functions $g_i^{p/n}$ are not very different from unity, and they were discussed above in (107).

If the target nuclei are not polarized, the terms proportional to ΔA^f in (128) disappear in the observed cross section after averaging, which takes the (well-known) form

$$\begin{aligned} \frac{d\sigma_{\text{inc}}^{\nu/\bar{\nu}}}{dT_A} = & \frac{2G_F^2 m_A}{\pi} g_i \sum_{f=p,n} \left[1 - |F_f(q^2)|^2 \right] \\ & \times A^f \left\{ (g_{L/R}^f)^2 + (g_{R/L}^f)^2 (1-y)^2 - 2g_L^f g_R^f \frac{ym^2}{s-m^2} \right\}, \end{aligned} \quad (130)$$

modified by the nuclear structural factor $1 - |F_f(q^2)|^2$.

¹⁶In the explicit form:

$$\begin{aligned} \frac{d\sigma_{\text{inc}}^{\nu}}{dT_A} = & \frac{2G_F^2 m_A}{\pi} \sum_{f=p,n} F_f \left\{ A^f \left[(g_L^f)^2 + (g_R^f)^2 (1-y)^2 - g_L^f g_R^f \frac{2m^2 y}{s-m^2} \right] + (+\Delta A^f) [g_L^f - g_R^f (1-y)] \left[g_L^f + g_R^f \left(1 - y \frac{s+m^2}{s-m^2} \right) \right] \right\}, \\ \frac{d\sigma_{\text{inc}}^{\bar{\nu}}}{dT_A} = & \frac{2G_F^2 m_A}{\pi} \sum_{f=p,n} F_f \left\{ A^f \left[(g_R^f)^2 + (g_L^f)^2 (1-y)^2 - g_L^f g_R^f \frac{2m^2 y}{s-m^2} \right] + (-\Delta A^f) [g_R^f - g_L^f (1-y)] \left[g_R^f + g_L^f \left(1 - y \frac{s+m^2}{s-m^2} \right) \right] \right\}. \end{aligned}$$

Averaging over spin in the expression for the coherent cross section zeros out the contributions proportional to ΔA_f , and after the opening of the square of the modulus formula (129) takes the form

$$\begin{aligned} \frac{d\sigma_{\text{coh}}^{\nu/\bar{\nu}}}{dT_A} = & \frac{G_F^2 m_A}{\pi} \left(1 - \frac{T_A}{T_A^{\text{max}}}\right) g_c \sum_{f,f'} F_f F_{f'}^* \left\{ g_V^f g_V^{f'} \right. \\ & \times \left[A_f A_{f'} \left(1 - \frac{y\tau}{2}\right)^2 + \Delta A_f \Delta A_{f'} \left(\frac{y}{2}\right)^2 \right] + g_A^f g_A^{f'} \\ & \times \left[A_f A_{f'} \left(\frac{y\tau}{2}\right)^2 + \Delta A_f \Delta A_{f'} \left(1 - \frac{y}{2}\right)^2 \right] + 2g_V^f (\pm g_A^{f'}) \\ & \times \left. \left(A_f A_{f'} \left(1 - \frac{y\tau}{2}\right) \frac{y\tau}{2} + \Delta A_f \Delta A_{f'} \frac{y}{2} \left(1 - \frac{y}{2}\right) \right) \right\}. \end{aligned} \quad (131)$$

Finally, expressions (130) and (131) can be considerably simplified by making the following approximations. First, the terms proportional to (the small value of the variable) $y \approx 3\% E_\nu / (30 \text{ MeV})$ and proportional to $\Delta A_f \Delta A_{f'}$ are omitted. The latter is quite valid for spin-zero nuclei or for heavy nuclei with $\Delta A \ll A$. Second, the role of the kinematic corrections is ignored by taking $g_{c/i} \rightarrow 1$. As a result, rather simple formulas are obtained for the coherent and incoherent cross sections

$$\begin{aligned} \frac{d\sigma_{\text{coh}}}{dT_A} = & \frac{G_F^2 m_A}{\pi} \left(1 - \frac{T_A}{T_A^{\text{max}}}\right) \sum_{f=n,p} A_f^2 |F_f|^2 (g_V^f)^2, \\ \frac{d\sigma_{\text{incoh}}}{dT_A} = & \frac{2G_F^2 m_A}{\pi} \\ & \times \sum_{f=n,p} A_f (1 - |F_f|^2) ((g_L^f)^2 + (g_R^f)^2), \end{aligned} \quad (132)$$

from which it is evident that (in the abovementioned approximation) the $\nu A \rightarrow \nu A^{(*)}$ and $\bar{\nu} A \rightarrow \bar{\nu} A^{(*)}$ cross sections are indistinguishable.

In addition, the terms proportional to g_V^p can be omitted, since $g_V^p \ll 1$ in the Standard Model, which changes Eq. (131) to a well-known result for the cross section of the coherent neutrino–nucleus scattering [1, 7, 33, 35, 38, 43–49] proportional to the square of the number of neutrons in the nucleus

$$\frac{d\sigma_{\text{coh}}}{dT_A} \approx \frac{G_F^2 m_A}{\pi} \left(1 - \frac{T_A}{T_A^{\text{max}}}\right) |F_n|^2 (g_V^n)^2 N^2. \quad (133)$$

The role of corrections to this formula are discussed below in section 7.4.

Concluding this section, we stress that a new and unique result is obtained [19, 20], which is demonstrated by formulas (125) and (126), (127) or (128), (129) and consists in “smooth” transition of the (anti)neutrino–nucleus scattering cross section from the coherent regime (129) to the incoherent regime (128).

Indeed, elastic (coherent, in standard terms) interactions, which keep the nucleus in the initial quantum state, lead to quadratic, in terms of the number of nucleons, enhancement ($\propto A_f^2$) of the observed cross section and simultaneously to its dependence on the square of the nuclear proton and neutron form factors $|F_{p/n}(\mathbf{q})|^2$ normalized to unity.

On the other hand, the cross section for inelastic (or incoherent, in standard terms) processes, where the quantum state of the target nucleus changes, linearly depends on the number of nucleons ($\propto A_f$) and at the same time is proportional to the expression $(1 - |F_{p/n}(\mathbf{q})|^2)$. Both terms (128) and (129) of total cross section (125) depend on the same nuclear form factors $F_{p/n}(\mathbf{q})$.

If $\mathbf{q} \rightarrow 0$, the form factors $F_{p/n}(\mathbf{q}) \rightarrow 1$, and the contribution from the incoherent term (128) to the total cross section vanishes, while the coherent term (129) totally determines the observed cross section (125). In the opposite case of large transferred three-momenta $\mathbf{q}$, the form factors $F_{p/n}(\mathbf{q}) \rightarrow 0$, and the coherent cross section (129) does not contribute while the incoherent cross section (128) completely determines the observed total cross section (125). Obviously, in the region of intermediate transferred three-momenta $\mathbf{q}$, the coherent and incoherent contributions should be considered simultaneously. Domination of one or another term can be roughly estimated (see, for example, (132)) from a comparison of factors like $A_{p/n} (1 - |F_{p/n}(\mathbf{q})|^2)$ and $A_{p/n}^2 |F_{p/n}(\mathbf{q})|^2$. This problem is discussed from the practical point of view in section 7.2.

7. NUMERICAL RESULTS AND DISCUSSIONS

In this section, using as an example a few typical target nuclei employed for low-threshold measurements, in particular for detecting and studying coherent neutrino–nucleus scattering, we will discuss the results obtained above and numerically estimate relative importance of contributions from coherent and incoherent processes to the total cross section of the (anti)neutrino scattering off these nuclei.

7.1. Three Typical Detectors and Nuclear Form Factors

From the practical point of view, it is convenient to deal with the cross section of the process under study

Fig. 11. Helm nuclear form factor F^{Helm} [55] for three target nuclei as a function of the absolute value of the transferred momentum $|q|$ (lower horizontal axis). The upper horizontal axis is the kinetic recoil energy of the ^{133}Cs nucleus corresponding to the given $|q|$.

integrated over the allowed interval of the kinetic energy of the recoil nucleus

$$\sigma(E_{\nu/\bar{\nu}}, T_A^{\min}) = \int_{T_A^{\min}}^{T_A^{\max}} \frac{d\sigma^{\nu/\bar{\nu}}}{dT_A} dT_A. \quad (134)$$

This integral cross section largely depends on the detection threshold $T_A^{\min}$ for the kinetic energy of the recoil nucleus (lowest recoil energy that can be detected) of each particular detector.

To perform this analysis, we take as an example three experimental facilities with the best energy thresholds in their class [19].

The first facility is the germanium detector normally used to detect $\bar{\nu}_e$ flux from nuclear reactors. Its working material is taken (for simplicity) to be only natural germanium isotopes ^{74}Ge . The expected detection threshold of germanium bolometric detectors for energy of electrons (from interaction with $\bar{\nu}_e$) is 200 eV [50], which, according to the estimation of the quenching factor for germanium detectors [51], corresponds to approximately 1 keV for the recoil kinetic energy of ^{74}Ge nuclei. An example of using this type detector is the νGEN experiment conducted by the JINR–ITEP collaboration at the Kalinin Nuclear Power Plant [52]. Differential cross sections for this detector are calculated at the (anti)neutrino energy $E_{\bar{\nu}} = 5$ and 8 MeV, and total cross sections are calculated for the interval of possible antineutrino beam energies $E_{\bar{\nu}} \in (1, 20)$ MeV. As an estimate of the lowest excitation energy of the ^{74}Ge nucleus, the value $\Delta\epsilon = 900$ keV was taken.

The second facility is the scintillation CsI detector (calorimeter) used by the COHERENT collaboration for detecting neutrinos from the Oak Ridge neutron source [8]. Differential cross sections for this facility are calculated at noticeably higher neutrino energies $E_{\nu 0} = 30$ and 50 MeV. Estimation of total cross sections is given for the interval $E_{\nu} \in (1, 150)$ MeV. It is assumed that the excitation energy of the ^{133}Cs nucleus is about $\Delta\epsilon = 100$ keV. The recoil energy threshold for the ^{133}Cs nucleus is 5 keV.

The third facility is the liquid-argon detector with an unprecedentedly low recoil energy detection threshold of 0.6 keV for ^{40}Ar nuclei reached in the DarkSide experiment [53]. Differential and total cross sections for this facility are obtained for the neutrino energies $E_{\nu} = 15(20)$ MeV and $E_{\nu} \in (1, 50)$ MeV respectively.

Numerical results were obtained using two (for protons and neutrons) nuclear form factors $F_{p/n}(\mathbf{q})$ and the data on excitation energy levels of the target nuclei studied. Two (most popular) parametrizations of nuclear form factors on the basis of the symmetrized Fermi distribution [54] and the Helm nuclear form factor [55] were considered. With the parameters of these models chosen under the requirement that the RMS radii of the proton and the neutron are equal, almost identical results were obtained for the nuclear form factors under consideration.¹⁷

Further, as in [19, 20], results are presented, which were obtained on the assumption of equal RMS proton and neutron radii and with the use (for definiteness) of the Helm nuclear form factors. Figure 11 shows the behavior of the Helm form factor as a function of the three-momentum $|\mathbf{q}|$ transferred to the nucleus for the three chosen nuclei. It is seen that for cesium nuclei at the nuclear recoil kinetic energy $T_A \approx 12$ –15 keV, where the peak of the signal observed by the COHERENT collaboration is located, the transferred momentum is $|\mathbf{q}| \approx 50$ –60 MeV/c, and the form factor squared is $|F(\mathbf{q})|^2 \approx 0.6$ –0.5, which clearly indicates noticeable suppression of the pure coherent contribution and presence of the contribution from incoherent processes (nuclear excitation).

7.2. Balance between the Coherent and Incoherent Terms

According to the conclusion in section 6.4, the most important (and new) property of the result obtained in [19, 20] and discussed here is a smooth transition of the (anti)neutrino–nucleus scattering cross section from the coherent to the incoherent regime, which is explicitly presented in formulas (125)

¹⁷ More details about these form factors are given in appendix 8.1.2.

Fig. 12. Differential cross sections $\frac{d\sigma^{\nu/\bar{\nu}}}{dT_A}$ versus the nuclear recoil energy T_A for coherent (solid lines) and incoherent (dashed lines) ν -nucleus (left column) and $\bar{\nu}$ -nucleus (right column) scattering for ^{74}Ge (top plots), ^{133}Cs (middle plots), and ^{40}Ar (bottom plots) target nuclei for different energies of initial (anti)neutrinos. Vertical lines correspond to the experimental energy thresholds for each of the targets.

and (128), (129) and in the simplified form in (132). The actual “regulator” of this transition is the nuclear nucleon form factors $F_{p/n}(\mathbf{q})$ defined in the general form by (108) and shown for illustration (in the Helm model [55]) in Fig. 11. This figure explicitly demonstrates that, as was already shown from general considerations in section 6.4, the form factor $F(\mathbf{q}) \rightarrow 1$ when $\mathbf{q} \rightarrow 0$, which results in that the contribution from the incoherent cross section disappears, and the coherent

term entirely dominates. On the other hand, at large $\mathbf{q}$ the form factor $F(\mathbf{q}) \rightarrow 0$, and now the coherent term disappears while the incoherent scattering cross section entirely dominates.

Let us see how it is practically realized. Figures 12 and 13 show differential and total (integral in terms of definition (134)) cross section for the coherent neutrino and antineutrino scattering off the nuclei at the abovementioned experimental facilities.

It is evident from the comparison of the left and right panels of the figures that the coherent and incoherent neutrino–nucleus and antineutrino–nucleus cross sections calculated using (128) and (129) demonstrate, in accordance with (132), almost identical behavior as a function of the recoil energy T_A and the incident (anti)neutrino energy E_ν , with only one insignificant difference for the heavy ^{133}Cs nucleus with its nonzero total spin. Next, as regards the balance of the coherent and incoherent cross sections, the neutrino and antineutrino beams have common properties.

If the nuclear recoil energy tends to zero, $T_A \rightarrow 0$, the coherent cross section dominates and fully governs the total cross section for the νA and $\bar{\nu} A$ scattering (125), since there is no incoherent contribution at all. For a particular nucleus, the differential cross section in this limit (up to insignificant corrections, according to (129)) does not depend on the (anti)neutrino energy.

If the nuclear recoil energy tends to its kinematic limit, $T_A \rightarrow T_A^{\max}$, the coherent cross section decreases rapidly and vanishes because it is proportional to the factor $1 - T_A/T_A^{\max}$, while the incoherent cross sections continuously increases with increasing T_A . Note that because the energy acquired by the nucleus can be used for excitation, the maximum achievable nuclear kinetic energy T_A in incoherent processes turns out to be systematically lower than the maximum achievable recoil energy $T_A^{\max}$ in the coherent case (see formula (60)). Therefore, the dashed (“incoherent”) curves in Fig. 12 always stop rising with increasing T_A somewhat earlier than the end point for the solid (“coherent”) curves is reached.

At low energies $E_{\nu/\bar{\nu}}$ the coherent cross section is always substantially larger than the incoherent cross section for any nuclear recoil kinetic energy T_A . At higher $E_{\nu/\bar{\nu}}$ the recoil energy T_A may already have values above which the incoherent cross section becomes larger than the coherent cross section, as is seen for $E_{\nu/\bar{\nu}} = 50$ MeV in the middle panels in Fig. 12. In particular, this takes place with the ^{133}Cs at $T_A \gtrsim 30$ keV.

At low beam energies $E_{\nu/\bar{\nu}}$ the integral coherent cross section (Fig. 13) is larger than the incoherent cross section by a few orders of magnitude since the factors $1 - |F_{p/n}(\mathbf{q})|^2$ strongly suppress the incoherent contribution at small transferred three-momenta $\mathbf{q}$. However, as the (anti)neutrino energy increases, this relation of the suppressive coherent contribution domination changes, and at certain $E_{\nu/\bar{\nu}}$ the integral incoherent cross section becomes rather appreciable.

The real experimental $T_A^{\min>0}$ detection threshold noticeably decreases integral cross sections with this effect being weaker for the incoherent scattering. The

nonzero threshold (vertical lines in Fig. 10) actually ignores a considerable part of the differential cross section in the region where this cross section makes the maximum contribution in the case of coherent scattering, as is seen in Fig. 12.

Figure 14 illustrates this statement quantitatively. It shows the ratio of the integral cross sections $\sigma_{\text{incoh}}/\sigma_{\text{coh}}$ for the heavy ^{133}Cs nucleus, which are calculated in accordance with (134). It is seen that for the νA scattering (left panel) at the neutrino energy $E_\nu \approx 30(60)$ MeV this ratio is approximately 7 (20)% when $T_A^{\min} = 0$ and reaches 15 (30)% for the real energy threshold $T_A^{\min} = 5$ keV. In this case, the incoherent contribution becomes equal to the coherent contribution when the incident neutrino energy is $E_\nu \approx 110$ MeV. Obviously, the role of incoherent interactions increases with increasing (incident neutrino) interaction energy.

Coherent and Incoherent Cross Sections

for $g_A^p = 1/2$ and $g_A^p = 1.27$

This section includes auxiliary figures that compare ultimate results for the coherent and incoherent neutrino–nucleus scattering obtained at different axial nucleon coupling constants $g_A^p = 1/2$ and $g_A^p = 1.27$, in particular, in [19]. Figures 15 and 16 show differential and integral cross sections for the coherent and incoherent neutrino–nucleus scattering, respectively. By and large, due to going from $g_A^p = 1/2$ to $g_A^p = 1.27$, the difference in the cross section curves is almost imperceptible for relatively light nuclei, such as ^{74}Ge (upper panels) and ^{40}Ar (lower panels).

Figure 17 demonstrates, by the example of the ^{133}Cs nucleus, how the value of the axial nucleon coupling constant g_A^p affects the incoherent-to-coherent cross-section ratio. It is seen that calculations with a more realistic axial nucleon coupling constant $g_A^p = 1.27$ lead to the increase of the role of the incoherent contribution with increasing neutrino energy in the case of a rather heavy ^{133}Cs nucleus, which, in addition, has a nonzero total spin.

7.3. Incoherent Interaction as a Source of Higher-Energy γ Rays

It is clear that after its interaction with neutrinos or antineutrinos with the discussed energies the nucleus has only two possible ways to follow: either it remains in the same quantum state (elastic scattering) or its internal quantum state changes (inelastic scattering). However, if the experimental facility is capable of

Fig. 13. Integral cross sections $\sigma^{\nu/\bar{\nu}}$ versus the incident (anti)neutrino energy for coherent (solid lines) and incoherent (dashed lines) ν -nucleus (left) and $\bar{\nu}$ -nucleus (right) scattering for ^{74}Ge (top), ^{133}Cs (middle), and ^{40}Ar (bottom) target nuclei. The cross sections are calculated both for idealized experimental facilities with the zero energy threshold ($T_A^{\min}=0$, blue curves) and for real facilities with their best energy thresholds $T_A^{\min}$ (red curves).

measuring only the kinetic energy of the recoil nucleus, it is almost impossible to know whether the nucleus remained in its initial state after the interaction or a transition to the excited level occurred (the state of the nucleus changed).

If the transition to the excited state is energetically possible, the resulting inelastic interaction should be accompanied by emission of photons arising from deexcitation returning the nucleus to the ground state. The energy of these photons will be determined by the

difference of the energy levels of the given nucleus [21]. For example, the time scale of emission of these gamma rays for the ^{133}Cs nucleus ranges from picoseconds to nanoseconds, and their energies are in the region of several hundreds of keV. Clearly, these photons (caused by deexcitation of the target nucleus) should produce a detectable signal in the scintillation detector-calorimeter, and this signal should be in general correlated with the time of target exposure to the accelerator neutrino beam.

Fig. 14. Coherent-to-incoherent cross section ratio $\sigma_{\text{inc}}/\sigma_{\text{coh}}$ as a function of the energy $E_{\nu/\bar{\nu}}$ for the ν scattering (left) and scattering (right) off the ^{133}Cs nucleus. The curves correspond to $T_A^{\text{min}=0}$ and the real (anti)neutrino detection threshold of 5 keV.

The counting rate of these γ rays is defined by the ratio $N_{\text{inc}}/N_{\text{coh}}$, where

$$N_{\text{inc/coh}} = \int dE_{\nu} \Phi(E_{\nu}) \int_{dT_A^{\text{min}}}^{dT_A^{\text{max}}} dT_A \frac{d\sigma_{\text{inc/coh}}}{dT_A} \varepsilon(T_A),$$

and $\varepsilon(T_A)$ is the detector efficiency. According to Fig. 14, it appears feasible to detect a noticeable number of events with γ rays from the incoherent interaction of (anti)neutrinos with the cesium nucleus (excitation of this nucleus).

Figure 14 is quite instructive. It follows from it that the COHERENT experiment (with the ^{133}Cs target) did see a noticeable part of the coherent elastic neutrino–nucleus scattering (CEvNS) but with a 15–20% uncertainty arising both from the high (anti)neutrino beam energy and from the facility’s high energy threshold of 5 keV. Figure 14 shows that the inelastic (incoherent) fraction of about 15–20% is unavoidable in the measured data of this experiment if γ rays from deexcitation of the nucleus are undetectable. Therefore, it is evident that any sufficiently careful analysis of the data from an experiment like COHERENT (e.g., in [56–66]) should consider this noticeable contribution from the incoherent scattering.

To our mind, there are two approaches to precision studies of CEvNS and its consequences. One is to separate the “true coherent and true elastic” signal from the incoherent contribution using the abovementioned possibility of observing deexcitation photons, which was first mentioned as far back as 1975 [21] and also recently, in the context of coherent neutrino scattering, in [19, 20].

Indeed, incoherent (inelastic) processes, being, according to Fig. 12, a relatively small background for coherent interactions (in the kinematic region under consideration), may give rather important information to an experimenter if there is a possibility of

detecting γ rays emitted by the nucleus excited due to interaction.

For each particular target nucleus, these photons will be characterized by three important parameters. First, their energy is usually noticeably higher than the recoil nucleus energy ($T_{\text{Cs}} \leq 40$ keV in Fig. 10) and is characterized by the difference of the nucleus excitation levels (up to 900 keV for Cs-133). Second, emission of photons with the energy determined by the difference of the nuclear excitation levels will be shifted in time (relative to the initial neutrino beam) by the times completely characteristic of the given nucleus for removing the given excitation energy. Third, the counting rate of the photons under discussion will be determined by the abovementioned ratio $N_{\text{inc}}/N_{\text{coh}}$.

Therefore, simultaneous (more exactly, correlated) detection of two signals in the form of the recoil energy of a nucleus and the deexcitation photon energy seems to be the only way to separate pure CEvNS from “secondary” inelastic processes and, consequently, to carry out precise CEvNS-based investigation of the nuclear structure and to search for new physics signals, as is extensively discussed [56–69].

Note that there is another way to study CEvNS using neutrino and antineutrino beams with much lower energies (compared to the COHERENT experiment), e.g., from nuclear reactors or from the Sun. Originally, long before the COHERENT ideas and results, it was regarded as the only way based on the striving to create and employ (anti)neutrino detectors with an extremely low energy detection threshold for nuclear recoil (below a hundred eV). They were supposed to be used for data taking in the regions where incoherent (inelastic) interaction is almost impossible (kinematically) or strongly suppressed, that is, where form factors $|F_{p/n}(\mathbf{q})|^2$ are almost indistinguishable from unity (see, for example, [61]). From our point of

Fig. 15. Differential cross sections $\frac{d\sigma}{dT_A}$ for the coherent (solid lines) and incoherent (dashed lines) scattering off the ^{74}Ge (top), ^{133}Cs (middle), and ^{40}Ar (bottom) nuclei for different neutrino energies. Vertical lines correspond to experimental thresholds. The left column is obtained at $g_A^p = 1/2$ and the right one is obtained at $g_A^p = 1.27$.

view, due to already achieved extremely low energy thresholds, most promising are the JINR–ITEP vGeN project [50, 52] at the Kalinin Nuclear Power Plant and, probably in the very near future, the Dark-Side experiment [53] in Gran Sasso. Other possibilities are discussed, for example, in [61].

7.4. Inaccuracies of the “Universally Adopted” Expression for the Coherent Cross Section

It is interesting to compare expressions for the coherent neutrino–nucleus scattering cross section (129) and (131) with the one generally used in the literature, e.g., in [10]

$$\begin{aligned} \frac{d\sigma_{\text{coh}}^0}{dT_A} &= \frac{G_F^2 m_A}{2\pi} \left[(G_V + G_A)^2 + (G_V - G_A)^2 (1 - y)^2 - (G_V^2 - G_A^2) \frac{m_A T_A}{E_V^2} \right] \\ &\approx \frac{G_F^2 m_A}{\pi} \left[G_V^2 \left(1 - \frac{T_A}{T_A^{\text{max}}} \right) + G_A^2 \left(1 + \frac{T_A}{T_A^{\text{max}}} \right) \right] \approx \frac{G_F^2 m_A}{\pi} \left(1 - \frac{T_A}{T_A^{\text{max}}} \right) |F_n|^2 (g_V^n)^2 N^2, \end{aligned} \quad (135)$$

Fig. 16. Integral cross sections of the coherent (solid lines) and incoherent (dashed lines) scattering off the ^{74}Ge (top), ^{133}Cs (middle), and ^{40}Ar (bottom) nuclei as a function of the neutrino energy. The cross sections are calculated both for idealized experimental facilities with the zero energy threshold ($T_A^{\min}=0$, blue curves) and for real facilities with their best energy thresholds $T_A^{\min}$ (red curves). The left column is obtained at $g_A^p = 1/2$, and the right one is obtained at $g_A^p = 1.27$.

where $G_V = F_p Z g_V^p + F_n N g_V^n$ and $G_A = F_p \Delta Z g_A^p + F_n \Delta N g_A^n$. The first approximate equality (second line) in (135) is a result of a rather accurate approximation $y = T_A/E_\nu \rightarrow 0$. The last line is the consequence of further simplifications: it is taken, first, that $g_V^p \rightarrow 0$ and, second, that the nucleus has the zero total spin.

With the above approximations, the third line in (135) would seem to fully coincide with the above-

derived approximate formula for the coherent neutrino–nucleus cross section (133). However, derivation of (135) appears to have nothing in common with the concept of coherence, since, as discussed above, coherent scattering corresponds to the case where after the interaction with the neutrino the nucleus (and correspondingly the nucleon with which the interaction took place) remains in its initial quantum state.

The structure of the first line in (135) is very similar to the expression obtained above for the incoherent

Fig. 17. Ratio $\sigma_{\text{inc}}/\sigma_{\text{coh}}$ of the incoherent νCs cross section to the coherent νCs cross section as a function of the energy E_ν . The curves correspond to $T_A^{\text{min}}=0$ and the real neutrino detection threshold of 5 keV. The left panel corresponds to $g_A^p = 1/2$, and the right one corresponds to $g_A^p = 1.27$.

cross section of the νA scattering, in which the nucleus changes its spin state (130),

$$\frac{d\sigma_{\text{inc}}^\nu}{dT_A} = \frac{2G_F^2 m_A}{\pi} g_i \sum_{f=n,p} \left[1 - |F_f(q^2)|^2 \right] \times A^f \left\{ (g_L^f)^2 + (g_R^f)^2 (1-y)^2 - 2g_L^f g_R^f \frac{ym^2}{s-m^2} \right\}$$

with the replacements $g_{L/R} = \frac{1}{2}(g_V \pm g_A)$ and $g_i [1 - |F_f|^2] = |F_f|^2$. The coincidence becomes simply “magic” if $F_p = F_n$, $Z = N$, and $\Delta Z = \Delta N$, more specifically, $Z = N = \Delta Z = \Delta N$.

The solution of the paradox why “coherent cross section” (135) coincides to a good accuracy with the incoherent one (128) and further with (130) is explained by deriving (135) with the use of the nonrelativistic approximation and summation over all spin projections of the active nucleon in the final state (i.e., neglect of the coherence condition).

Indeed, two terms in the matrix element, which are proportional to the scalar products of the currents (l, h_{+-}^η) and (l, h_{-+}^η) corresponding to the change in the projection of the nucleon spin on the chosen axis, should not contribute to the coherent cross section (since they change the nuclear spin), but they are explicitly present in (135). In the nonrelativistic approximation, the first scalar (l, h_{+-}^η) is zero while the second scalar (l, h_{-+}^η) is proportional to g_A and vanishes for spin-zero nuclei, as is evident from (301). The latter is valid if the nucleons in the nucleus are at rest (i.e., the nonrelativistic approximation applies).

To illustrate the significance of effects related to the motion of the target nucleus and the constancy of

the nuclear spin in the elastic neutrino–nucleus scattering, Fig. 16 shows ratios of differential coherent cross sections $d\sigma/dT_A$ for the ^{133}Cs nucleus calculated according to (129) and (135) for three neutrino beam energies. It is seen that these cross sections coincide at the zero nuclear recoil energy $T_A = 0$ and begin showing a difference at a level of few percent as T_A increases. The difference reaches its maximum at the end of the spectrum of the allowed nuclear kinetic energy, increasing with energy from 5% at $E_\nu = 30$ MeV to 20% at $E_\nu = 100$ MeV.

Thus, there are two causes for this difference. First, in the case of coherent interaction, the above general formulas are derived by considering only those nucleon matrix elements, which correspond to the same initial and final spin state of the nucleon (nucleus). This derivation contradicts the canonical derivation of the formula for coherent scattering (used in [10]) where contributions with nucleon spin flip are explicitly considered as well. Second, the target nucleon is not assumed to be at rest (in the nucleus), which leads to corrections in the vector and axial nuclear form factors.

Thus, the use of approximate formula (135) to describe and analyze the CEvNS data from an experiment like COHERENT may entail another systematic error at a level of 5–15%.

7.5. Possibility of Nucleon Spin Flip in Coherent Scattering

The above approach to a unified description of the coherent and incoherent neutrino and antineutrino scattering off a nucleus is based on [19] and proceeds from first principles. Obviously, the analytical formu-

Fig. 18. Ratio R of the differential coherent cross section $d\sigma/dT_A$ calculated by formula (129) to the differential coherent cross section used by the COHERENT collaboration and presented by Eq. (135). Both cross sections are averaged over the nuclear spin and correspond to the neutrino scattering off the ^{133}Cs nucleus. The ratio is given as a function of the nuclear recoil kinetic energy T_A .

las for description of the (anti)neutrino scattering off the nucleon that is a constituent of a complex bound system (nucleus) in [19, 20] could only be used with a number of simplifying approximations and assumptions.

In particular, the most subjectable to criticism seems to be the assumption made in [19] that no contribution comes to the coherent (or elastic) cross section from the amplitudes (or, in our case, scalar products) associated with a possibility of the spin flip of the nucleon in the events of incident (anti)neutrino scattering off it. It appears to be more or less understandable that the spin flip at the level of the nucleon may well be “imperceptible” at the level of the nucleus. The projection of the nuclear spin (on the preferred direction) might not change, since there is no denying that the spin flip of one nucleon can be compensated by the corresponding spin flip of another nucleon. Therefore, the nucleon spin flip seems somewhat possible for both inelastic and elastic scattering. However, the question is still open as to the characteristic time that “another nucleon” needs to adapt to the “first nucleon” so that the nucleus containing them finds itself in the initial state again and how this time of “nuclear spin recovery” is related to the characteristic time of deexcitation of this nucleus by its returning from the excited level to the initial state.

In other words, the question is, how much time should pass since the instant of interaction for the process to be considered elastic (the nucleus remained in the initial state or returned to it so fast that no signals

about possible temporary change in the state were received) or inelastic (the nucleus returned to the initial through emission of a photon resulting from its deexcitation)?

On the other hand, probabilities of such spin flips are determined by the nuclear wave function that can now hardly be obtained from the same first principles which underlay the approach used in [19].

Moreover, any spin flip of a particular nucleon in a target nucleus with the nonzero total spin must necessarily (at least for some time) change the total nuclear spin and consequently cause excitation of this nucleus. Of course, this excitation is sooner or later removed by subsequent “disappearance of the induced” nuclear spin and the energy release. Therefore, nuclei with the total nuclear spin noticeably smaller than the number of the nucleons themselves, $A_{p/n} \gg \Delta A_{p/n}$, will most probably behave like spin-zero nuclei, i.e., any spin flip of a particular nucleon in the target nucleus will lead to a change in the energy state of the entire target nucleus, that is, to the inelastic nuclear excitation process. For these reasons, neglect of the contribution from events with nucleon spin flips to the cross section of the coherent (anti)neutrino–nucleus scattering corresponds well, from our point of view, to the level of accuracy of the consideration.

7.6. (In)coherent Scattering or (In)elastic Scattering?

According to the tradition dating back to Freedman’s pioneering work, terms (128) and (129) of the total (anti)neutrino–nucleus scattering cross section (125) are denoted by the indices “coherent” and “incoherent” respectively. In fact, this terminology is also used through this entire text.

Nevertheless, as shown above, for neutrino and antineutrino energies in the region of tens of MeV terms (128) and (129) are coherent in the sense that all nucleons of the nucleus are (almost) equally involved in the interaction at the level of scattering amplitudes [19]. It is this property that was constructively used to obtain key expressions for the (anti)neutrino–nucleus scattering cross section (125) and (128), (129). Therefore, strictly speaking, it is more correct to use, at least in the kinematic region under discussion, the terms (indices) “elastic” and “inelastic” scattering/interaction instead of “coherent” and “incoherent”.

At considerably higher energies (tens of GeV and higher) it seems reasonable to generalize, at least qualitatively, the notion of completeness of nuclear excitation energy levels used in this consideration for the $\nu A \rightarrow \nu A^*$ process to the case where the target nucleus is completely destroyed, $\nu A \rightarrow \nu X$ (deep inelastic scattering). Obviously, complete destruction of the nucleus means that the notion of its nucleon form factors becomes meaningless, which is equivalent to

$F_{p/n}(\mathbf{q}) \equiv 0$. Therefore, the inelastic (incoherent) cross section for the (anti)neutrino–nucleus scattering in the form (128) becomes very similar to the main result of the parton model with its famous linear dependence on the total number of nucleons in the nucleus A .

A possible further development (generalization) of the proposed approach can be through using, in addition to the nucleon nuclear form factors $F_{p/n}(\mathbf{q})$, the proton and neutron form factors $f_{p/n}(\mathbf{q})$ that allow for the internal proton and neutron structure, e.g., in the form of replacement of chiral nucleon weak coupling constants $g_{L/R}^{p/n}$ with expressions depending on the transferred momentum like $g_{L/R}^{p/n} f_{p/n}(\mathbf{q})$.

Another possible way to generalize the considered approach [19, 20] can be by using wave packets instead of plane waves to describe fermions (see, for example, [70–72]). This generalization is of particular interest from the point of view of new physical phenomena in the scattering of neutrinos and antineutrinos off nuclei.

8. PHYSICAL PROBLEMS AND EXPERIMENTS ON CEvNS

8.1. Physical Problems for Coherent Neutrino Scattering

In this section solutions of some physical problems currently related to the study of CEvNS are briefly reviewed.

8.1.1. Role of CEvNS in supernova explosions. The one to begin with is Freedman’s work of 1973 [1], where the role of a considerable increase in the CEvNS cross section was first discussed in the context of explaining a supernova explosion. Next, in 1974, Wilson proposed [73] a new stellar collapse mechanism based on coherence, “discovered” by Freedman, and its related proportionality of the cross section for neutrino scattering off heavy nuclei (neutral current) to the square of the atomic weight. According to [73], collapse of a star with an iron core leads to formation of a hot neutron star surrounded by a thin iron layer. Neutrinos produced in the hot neutron core are able, due to a large cross section of their interaction in the external (iron) layer, to produce a pressure at the iron layer accelerating it to escape velocities and thus explode the star.

In [74], interesting arguments are given in favor of the point that consideration of the nuclear temperature of the star can be an important factor for the increase in cross sections for neutrino scattering off excited states of nuclei inside the star. On this basis, cross sections were calculated for the neutrino–nucleus scattering that refer to processes in supernovas [74].

According to contemporary ideas, CEvNS plays an important role in the evolution of a supernova—it “sets off” the supernova explosion, i.e., a powerful

neutrino flow from the star core blows with a great force the outer layer of the star off into space, which is observed as a supernova (SN).

Moreover, current models of SN evolution include this process as the main source of opaqueness for neutrinos which is of particular importance in the star collapse phase before the SN explosion [75]. During the collapse, the core of the massive star is rapidly deleptonized, i.e., absorbs electrons with emission of neutrinos and grows enriched in neutrons. As more and more of these neutrinos are captured (due to CEvNS), the deleptonization rate decreases. Depending on the magnitude of CEvNS, the moment in which neutrinos are captured changes. In particular, a greater magnitude of CEvNS causes neutrino capture at a higher electron fraction while a smaller magnitude of CEvNS leads to neutrinos being captured later at a lower electron fraction. As a result, a greater magnitude of CEvNS yields a more massive nucleus, and a smaller magnitude of CEvNS yields a lighter nucleus. Then, the heavier nucleus gives rise to a shockwave, which first moves in the star but then stops at its relatively small radius. In contrast, the shockwave occurring in the case of the light nucleus first moves rather slowly and does not stop at the same small radius. Related to this effect is variation in the luminosity of emitted μ and τ neutrinos and antineutrinos. For example, a change in the CEvNS cross section by a factor of two leads to a change in the peak luminosity of ν_μ and ν_τ from the supernova by 10% [75].

A unique point is that the same CEvNS effect provides experimenters with a method for detecting and studying the flow of “mighty” neutrinos from the supernova explosion and allows verifying model ideas of the explosion mechanism.

Indeed, the use an ordinary (not CEvNS) detector for detecting neutrinos from the supernova has a disadvantage that in this detector the largest number of events from the galactic supernova is expected in the charged current channel, i.e., from the inverse β decay $\bar{\nu}_e + p \rightarrow n + e^+$. The goal of the measurements is the spectrum of neutrinos of different flavors, and as much as possible the same as was emitted by the neutron protostar from the center of the supernova. However, effects of neutrino oscillations, possible effects of neutrino self-action, and so on, will mix spectra of particular neutrino flavors on the escape of neutrinos from supernovas. Neutral-current reactions in the form of CEvNS are insensitive to this mixing, and the rate of these reactions is determined by the sum of neutrinos and antineutrinos of all three flavors. They can be used together with charged-current signals to separate flavor transformation effects from the initial neutrino signal. This makes CEvNS particularly important for observing supernovas and a key reaction for measuring the character of evolution of the neutrino flux from a supernova.

Thus, observation of galactic neutrinos from supernovas is one of the most important current objectives for neutrino physics and astrophysics of rather low energies. The key role in attaining this objective is played by CEvNS (see, for example, [47, 76, 77]).

8.1.2. CEvNS and the nuclear structure. Since it is the complex nuclear structure that makes CEvNS particularly significant in solving problems in the field of neutrino physics and astrophysics, adequate knowledge of this structure in the form of models of the nucleus and quantitative predictions on their basis are obviously crucial in using CEvNS for solving fundamental problems (detection of supernova bursts and so on) and for practical purposes (monitoring of industrial nuclear reactors and so forth).

An example of this kind of calculation on the basis of present-day nuclear models is given in [38, 78]. The calculations were made to find both total cross sections and expected event counting rates for the coherent and incoherent neutrino scattering (neutral current channel) off ^{40}Ar and ^{132}Xe isotopes at neutrino energies $E < 100$ MeV. The coherent channel was described by the transition $0_{gs}^+ \rightarrow 0_{gs}^+$ that was significantly dominant over the incoherent one to which the nuclear transition $0_{gs}^+ \rightarrow J_f$ corresponded. The expected response of the above isotopes to the possible neutrino signal from the SN was calculated using the Fermi–Dirac distribution for energy spectra of SN neutrinos [38]. Similar nuclear-model calculations of probabilities for detection of neutrino and antineutrino signals from a possible supernova explosion performed for other targets from ^4He to ^{208}Pb were presented in [78]. Note that in these works, it was assumed (without appropriate substantiation) that coherent elastic neutrino and antineutrino scattering off nuclei is described by the same formulas.

Another approach based on the theory of Fermi liquid was used in [79] to estimate cross sections for elastic and inelastic neutrino scattering off nucleons in an extended nuclear medium. The cross sections were calculated for neutrino scattering off heavy nuclei at energies below 50 MeV in terms of dynamic form factors. These cross sections agree with the results of other much more complicated models of the nucleus [79].

Thorough nuclear-model calculations of differential and total cross sections for neutrino scattering (neutral current channel) off the most abundant xenon isotope were carried out in [39]. The calculations were performed within the nuclear shell model for elastic scattering and within the quasiparticle random phase approximation (QRPA) and the microscopic quasiparticle–phonon model (MQPM) for both elastic and inelastic scattering. Average cross sections for solar ^8B neutrinos and supernova neutrinos were also estimated [39].

In [66], capabilities of the COHERENT experiment in determining and studying such an important characteristic of the nuclear structure as the neutron nuclear form factor were investigated. Nuclear form factors were defined in [66] as Fourier transforms of the densities of charge distributions in the nucleus

$$F_{n,p}(q^2) = \frac{1}{N_{n,p}} \int \rho_{p,n}(\mathbf{r}) e^{i\mathbf{q}\cdot\mathbf{r}} d^3\mathbf{r}, \quad (136)$$

where it was not assumed that $F_p = F_n$. These nuclear form factors can be written as an expansion in moments of charge density distributions [44]

$$F_{p,n}(q^2) \approx 1 - \frac{q^2}{3!} \langle R_{p,n}^2 \rangle + \frac{q^4}{5!} \langle R_{p,n}^4 \rangle - \frac{q^6}{7!} \langle R_{p,n}^6 \rangle + \dots, \quad (137)$$

where the k th moment is defined as

$$\langle R_{p,n}^k \rangle = \frac{\int \rho_{p,n}(\mathbf{r}) |\mathbf{r}|^k d^3\mathbf{r}}{\int \rho_{p,n}(\mathbf{r}) d^3\mathbf{r}}. \quad (138)$$

The authors [66] performed calculations within the current deformed shell model (DSM). The form factors of the ^{127}I and ^{133}Cs nuclei were calculated using the effective interaction [80] and the space of states of spherical orbitals $0g_{7/2}$, $1d_{5/2}$, $1d_{3/2}$, $2s_{1/2}$, and $0h_{11/2}$ on the basis of the filled ^{100}Sn core. Six internal configurations for ^{127}I and three for ^{133}Cs were considered. This was enough to adequately reproduce the most important characteristics of these isotopes.

A comparison was made of the DMS calculations and the widely used Helm [81], symmetrized Fermi [82], and Klein–Nystrand [83] form factor approximations.

As is known, the approximate Helm approach is based on the convolution of two nuclear densities, the uniform density with the cutoff radius R_0 (diffraction radius) characterizing the internal nuclear structure and the density associated with the Gaussian falloff at the edge of a nucleus and expressed in terms of the surface thickness s ($s = 0.9$ follows from the muon spectroscopy data)

$$F_{\text{Helm}}(q^2) = 3 \frac{j_1(qR_0)}{qR_0} e^{-(qs)^2/2}, \quad (139)$$

where $j_1(x) = \sin(x)/x^2 - \cos(x)/x$ is the spherical Bessel function of the first kind. In this case, the first three moments can be analytically expressed as [54]

Fig. 19. Weak proton and neutron form factors of the ^{133}Cs (left) and ^{127}I (right) nuclei as a function of the transferred momentum $q = Q$ (fm^{-1}) calculated in the DSM approach. For comparison, Helm, SF, and KN form factors are shown. Borrowed from [66].

$$\begin{aligned}\langle R_n^2 \rangle &= \frac{3}{5} R_0^2 + 3s^2, \\ \langle R_n^4 \rangle &= \frac{3}{7} R_0^4 + 6R_0^2 s^2 + 15s^4, \\ \langle R_n^6 \rangle &= \frac{1}{3} R_0^6 + 9R_0^4 s^2 + 63R_0^2 s^4 + 105s^6.\end{aligned}\quad (140)$$

Based on the universally accepted Fermi (or Woods–Saxon) charge density distribution, the symmetrized Fermi (SF) form factor, according to [82], has the form

$$\begin{aligned}F_{\text{SF}}(q^2) &= \frac{3}{qc[(qc)^2 + (\pi qa)^2]} \left[\frac{\pi qa}{\sinh(\pi qa)} \right] \\ &\times \left[\frac{\pi qa \sin(qc)}{\tanh(\pi qa)} - qc \cos(qc) \right],\end{aligned}\quad (141)$$

where the parameters expressed in fm

$$c = 1.23A^{1/3} - 0.60, \text{ and } a = 0.52, \quad (142)$$

have the meaning of the half-density radius and the diffuseness. The nuclear surface thickness is parametrized in the form of the relation $t = 4a \ln 3$ [84]. For the SF form factor, the first three moments in expansion (137) are, according to [54],

$$\begin{aligned}\langle R_n^2 \rangle &= \frac{3}{5} c^2 + \frac{7}{5} (\pi a)^2, \\ \langle R_n^4 \rangle &= \frac{3}{7} c^4 + \frac{18}{7} (\pi a)^2 c^2 + \frac{31}{7} (\pi a)^4, \\ \langle R_n^6 \rangle &= \frac{1}{3} c^6 + \frac{11}{3} (\pi a)^2 c^4 + \frac{239}{15} (\pi a)^4 c^2 + \frac{127}{5} (\pi a)^6.\end{aligned}\quad (143)$$

There is also the Klein–Nystrand (KN) form factor, which formally follows from the convolution of the

Yukawa potential with the dimensional parameter $a = 0.7$ fm and the Woods–Saxon distribution approximated by a rigid sphere with radius R_A . As a result [83], one gets

$$F_{\text{KN}}(q^2) = 3 \frac{j_1(qR_A)}{qR_A} \frac{1}{1 + (qa)^2}, \quad (144)$$

where the corresponding root-mean-square radius is

$$\langle R^2 \rangle_{\text{KN}} = \frac{3}{5} R_A^2 + 6a^2.$$

Figure 19 from [66] depicts DSM-calculated form factors and parametrizations of the Helm, SF, and KN form factors. It is seen that the approximation $F_p = F_n$ is not always good because the extrema of the proton form factors F_p and neutron form factors F_n correspond to different values of the transferred three-momentum.

In [66] it is shown that the calculations considering deformations of the nuclei under consideration noticeably better agree with the CEvNS data, because a considerable fraction of the theoretical uncertainty arises from the neutron nuclear form factor. It is very important to decrease this uncertainty to reliably analyze the data and obtain precision results [66].

The highest importance of the neutral-current coherent elastic neutrino–nucleus scattering for measuring and studying electroweak form factors for a large number of various nuclei and gaining information on the neutron distribution in the nucleus and generally on the neutron–nuclear structure has been pointed out many times in the literature (see, for example, [26, 44, 85]).

The measurements of the neutron density in nuclei are of interest from many points of view. In particular, the root-mean-square neutron radius R_n of a nucleus and the difference between R_n and the root-mean-square proton radius R_p (the so-called neutron shell) are the main components of the equation of state of nuclear matter (EOS). The latter plays a significant role in the understanding of some points, such as nuclear properties in laboratory experiments, features of heavy-ion collisions, properties of superheavy elements, and structure and evolution of compact astrophysical objects like neutron stars (see, for example, [84] and references therein).

In [84], the root-mean-square neutron radius of the ^{133}Cs and ^{127}I nuclei was obtained for the first time using the CEvNS data from the COHERENT experiment and symmetrized Fermi and Helm form factors (see (141) and (139)). The almost model-independent value of the radius was found to be $R_n = 5.5^{+0.9}_{-1.0}$ fm.

Recall that the root-mean-square (RMS) charge radius of the proton is defined (from the electron–proton scattering) by the condition [86]

$$R^2 = -6 \frac{G(q^2)}{dq^2} \Big|_{q=0},$$

where $G(q^2)$ is the electric proton form factor. This definition also applies to nuclear form factors. If the form factor is parametrized in the form of a series in q^2 (see, for example, expansion (137)), $G(q^2) = 1 - \frac{q^2 R^2}{6} + O(q^4)$, the precision data at quite small q^2 directly allow determining the term $\frac{q^2 R^2}{6}$, i.e., the quantity R .

For the Helm nuclear neutron form factor (139), the RMS neutron radius is given by the upper formula (140). Another important quantity is the difference of the RMS nuclear proton and neutron radii

$$\Delta R_{np} = R_n - R_p, \quad (145)$$

which is called the “neutron shell”. Information on this quantity is in rather high demand because ΔR_{np} correlates with a few properties characterizing neutron-rich matter (see references in [84]). A larger neutron shell implies a larger neutron star radius R_{NS} . Since the neutron star binding energy is inversely proportional to the radius R_{NS} of the star itself, then the larger R_{NS} , the lower the gravitational binding energy, which can be verified by observing an intense neutrino burst from the supernova [84]. The neutron shell is also related to a number of other nuclear quantities (see [87]).

In [84], the RMS proton radii of the ^{127}I and ^{133}Cs nuclei $R_p^{\text{Cs}} = \langle r_p^2 \rangle_{\text{Cs}}^{1/2} = 4.804$ fm and $R_p^{\text{I}} = \langle r_p^2 \rangle_{\text{I}}^{1/2} = 4.749$ fm are given, and on their basis, the neutron shell was estimated at $\Delta R_{np} \approx 0.7^{+0.9}_{-1.1}$ fm, which does not allow stating that it is different from zero. However, in 2018, using additional information on parity violation in atomic systems, the authors [56] obtained a more accurate result $R_n = 5.42 \pm 0.31$ fm. This value made it possible to calculate for the first time the cesium neutron shell $\Delta R_{np} = 0.62 \pm 0.31$ fm, which is already different from zero at a level of 2σ [56].

A similar analysis of the COHERENT data was made in [64], where slightly different values of the cesium neutron shell $\Delta R_{np} = 0.24$ fm and the weak Weinberg angle $\sin^2 \theta_W^* = 0.21$ were obtained, though with a considerable uncertainty, which the authors discussed from the point of view of possible New Physics effects [64].

Of special note is the result obtained in [84]. Actually, it was mentioned for the first time that the COHERENT data indicated suppression of the full coherence due to the nuclear structure at a level of 2.3σ . Further analysis made by the authors [88] with a more accurate value of the quenching factor from [67] increased the significance of the suppression of the full coherence to the level of 3.7σ . This result actually unambiguously confirms our [19] statement about importance of the incoherent component for analysis of the COHERENT data.

As was mentioned many times, the CEvNS cross section depends on the uncertainty in the nuclear form factor, which arises, to a first approximation, from inaccurate knowledge of the root-mean-square radius of the neutron density distribution in the nucleus. These uncertainties for cesium iodide, germanium, xenon, and argon detectors were thoroughly studied in [63]. They were shown to become substantial when the momentum transferred to the nucleus reaches the value $q > 20$ MeV/c and depend very little on the particular parametrization of the form factors of those nuclei. Uncertainties are larger for lighter elements, e.g., for argon they are 11%. The authors [63] showed that the uncertainties in the form factors were not important for CEvNS induced by reactor or solar neutrinos (by virtue of lower energies). Considering these uncertainties, the effect was estimated which they could produce on the accuracy of future COHERENT measurements of CEvNS induced by neutrinos from the diffuse supernova background (DSNB) and atmospheric neutrinos with energies below a GeV. It was shown that relative uncertainties in the number of detected events can amount to 50% depending on the recoil energy and the type of target. Uncertainties for DSNB and atmospheric neutrinos can be as large as 7 and 16%, respectively. It was also shown that the result of the search for New Physics using the CEvNS

Fig. 20. Illustration of the role of the neutrino floor in the direct detection of dark matter particles. Borrowed from [93].

data quite strongly depends on uncertainties in the nuclear neutron form factor, which, if incorrectly considered, can lead to misidentification of new physical signals [63].

During the preparation of this review new works were published that dealt thoroughly with the nuclear structure in neutrino–nucleus scattering processes [89, 90].

8.1.3. CEvNS and direct search for dark matter.

First of all, the relation between CEvNS and terrestrial experiments aimed at searching for dark matter (DM) arises from the striving to observe a signal from nuclear recoil caused by the interaction of the target nucleus with the neutrino or a DM particle in the detector of recoil energy. Therefore, if the goal of the experiment is the direct search for events from interaction of dark matter particles with the target material of a highly sensitive low-threshold detector (located deep under *Earth's* surface), any physical process that causes recoil of the target nucleus becomes an unavoidable background. Indeed, the behavior of recoil nuclei from CEvNS is now almost indistinguishable from the behavior of recoil nuclei after their (expected) interaction with massive neutral dark-matter particles (WIMPs).

Since the current trend to increase sensitivity of experiments on direct search for dark matter under terrestrial conditions is connected with attempts to shift down exclusion curves in the DM cross section–DM mass plane¹⁸, the progress in this direction will

obviously very soon “come across” these unavoidable signals (in the form of recoil nuclei) from solar and atmospheric (terrestrial, reactor, etc.) neutrinos (Fig. 20).

Therefore, the direct search for dark matter seems to have no future without additional information that allows the background CEvNS neutrino signal to be separated from the desired (true) WIMP signal in the dark matter detector. Or, to put it mildly, noticeable difficulties arise in interpretation of results of these experiments in terms of limits on the mass and the WIMP interaction cross section. Thus, CEvNS is the sensitivity limit for future (traditional) direct search for dark matter (see, for example, [47, 93]).

It is worth mentioning, however, that a competition for obtaining the best exclusion curve has been going on for many years, which involved almost all experiments aimed in one way or another at direct and indirect search for dark under terrestrial conditions. Unfortunately, this competition has very little to do with real possibility of observing DM particles. Real dark matter particles have an exclusive galactic signature, namely annual modulation of a signal, which is currently accessible only to direct DM detection experiments. This signature can in no way be observed using colliders and accelerators or in experiments on indirect searches for dark matter. Therefore, to prove that the DM candidate observed (if ever) in a collider is a true DM particle, it must be observed in an experiment on direct DM detection and its galactic nature has to be demonstrated [94]. Moreover, this exclusive galactic DM signature, the annular signal modulation, becomes a must for any DM experiment, since it now

¹⁸With few exclusions, e.g., the DAMA/LIBRA collaboration [91, 92].

appears to be the only real possibility of separating the DM particle interaction signal from the neutrino (background) CEvNS signal in any terrestrial DM detector [94].

However, an interesting possibility of avoiding (at least temporarily) this “threat” was proposed in [95], where expected DM and CEvNS signals were thoroughly simulated using the results of the analysis of achieved energy resolutions and thresholds, and also beta and gamma-ray backgrounds. The authors [95] asserted that the difference in the behavior of the CEvNS spectrum and the DM spectrum at DM particle masses above $15 \text{ GeV}/c^2$ and the exposure of 10 kg-yr and larger can make it possible to investigate WIMP–nucleon interaction below the neutrino CEvNS background. The best success can be achieved, according to [95], by reliably measuring the CEvNS spectrum from atmospheric and solar neutrinos (which requires exposure of 50 kg-yr) and treating this spectrum as an eliminable (subtractable) background [95].

The question of how large can be the CEvNS-caused neutrino background, or the neutrino “floor,” is very important for direct dark matter detection prospects. Dependence of the location of this neutrino floor on modern New Physics models in the neutrino sector was studied in [57]. In particular, it was found out that in the case of New Physics scalar interaction carriers (mediators), this neutrino floor can go up by at least an order of magnitude in the region of small dark matter masses (below $10 \text{ GeV}/c^2$). In the case of New Physics vector mediators, this increase can be as large as a factor of two. Therefore, as is stated in [57], the signal of direct dark matter particle detection in the region of low masses requires a particularly careful analysis of its reliable excess over the neutrino background.

The idea of using facilities originally intended for direct detection of dark matter particles to search for CEvNS was earlier considered, for example, in [96]. It is based on the fact that signatures (in recoil nucleus energy) of coherent elastic interaction of the neutrino and the WIMP nucleus will be very similar. It was estimated how the next-generation dark matter detector can detect coherent neutron–nucleus scattering, and so on [96].

On the contrary, in [97] the authors studied capabilities of experiments initially intended for CEvNS observation in detecting and investigating new light weakly interacting states, such as dark matter particles with a mass below $1 \text{ GeV}/c^2$ from various extensions of the Standard Model. It was shown that the proposed experiments on coherent neutrino scattering, such as COHERENT at SNS and CENNS at Fermilab, will have sensitivity to states of this sort noticeably above the currently existing limits [97].

In [98], the data of the TEXONO and COHERENT experiments were used to obtain new limits on the “photon portal”, in which the dark photon is kinematically mixed with the ordinary Standard Model photon. Electron recoil energy events in the TEXONO reactor-based experiment and nuclear recoil energy events in the COHERENT experiment were analyzed. The analysis showed that combined constraints give the mass limit below $30 \text{ MeV}/c^2$ for these dark matter particles [98].

A possibility that a new fermion (with an MeV mass) is created in coherent elastic neutrino–nucleus scattering was considered in [58]. This fermion could be a dark matter particle and/or be related to formation of light neutrino masses in the Standard Model. The impact of consideration of this effect on the measured nucleon recoil spectrum was calculated. Limits on the mass and parameters of coupling with neutrinos and quarks were obtained for this fermion from the COHERENT data [58].

8.1.4. Neutrino properties and CEvNS. It is commonly believed that neutrinos are neutral particles, but there are still no severe theoretical and experimental constraints preventing them from having a very small electric charge and (electric) charge radii (see, for example, [99]). In the theory of electromagnetic neutrino interactions, neutrino charge radii are defined for massive neutrino states [99]. Effective charge radii $\langle r_{\nu_\ell}^2 \rangle$ are also considered [59] for fixed-flavor neutrinos ν_ℓ , ($\ell = e, \mu, \tau$), whose contribution to elastic scattering of ν_ℓ by charged particles can be expressed as a shift of the Weinberg weak mixing angle θ_W . This shift is proportional to the square of the W boson mass m_W [37, 100]

$$\sin^2 \theta_W \rightarrow \sin^2 \theta_W \left(1 + \frac{1}{3} m_W^2 \langle r_{\nu_\ell}^2 \rangle \right). \quad (146)$$

According to [59], in the Standard Model there are predictions for effective charge radii of neutrinos determined by contributions of higher-order loop diagrams

$$\langle r_{\nu_\ell}^2 \rangle_{\text{SM}} = \frac{G_F}{2\sqrt{2}\pi^2} \left[3 - 2 \ln \left(\frac{m_\ell^2}{m_W^2} \right) \right],$$

where m_ℓ is the mass of the charged lepton. These predictions in the numerical form [59] are

$$\begin{aligned} & \{ \langle r_{\nu_e}^2 \rangle, \langle r_{\nu_\mu}^2 \rangle, \langle r_{\nu_\tau}^2 \rangle \}_{\text{SM}} \\ &= -\{0.83, 0.48, 0.30\} \times 10^{-32} \text{ cm}^2. \end{aligned} \quad (147)$$

In [59], the impact of neutrino charge radii on the cross section for coherent elastic neutrino–nucleus scattering [49, 101] was investigated and limits on these neutrino radii were obtained from the analysis of the COHERENT data [8, 11]. The approach [59] was based on the expression for the differential cross sec-

tion of the process $\nu_\ell + A \rightarrow \sum_{\ell'} \nu_{\ell'} + A$ for the case of the scattering off the spin-zero nucleus A in the approximation of smallness of the nuclear recoil kinetic energy $T_A \ll E_\nu$

$$\begin{aligned} \frac{d\sigma_{\nu_\ell A}(E_\nu, T_A)}{dT_A} &\approx \frac{G_F^2 M}{\pi} \left(1 - \frac{MT_A}{2E_\nu^2}\right) \\ &\times \left\{ \left[(g_V^p - \tilde{Q}_{\ell\ell}) Z F_Z(q^2) + g_V^n N F_N(q^2) \right]^2 \right. \\ &\quad \left. + Z^2 F_Z^2(q^2) \sum_{\ell' \neq \ell} |\tilde{Q}_{\ell'\ell}|^2 \right\}, \end{aligned} \quad (148)$$

where $\tilde{Q}_{\ell\ell} \frac{2}{3} m_W^2 \sin^2 \theta_W \langle r_{\nu_{\ell\ell}}^2 \rangle$ explicitly allows for the contribution from the nondiagonal neutrino effective charge radius $\langle r_{\nu_{\ell\ell}}^2 \rangle$, T_A in the kinetic energy of the nuclear recoil, E_ν is the neutrino energy, $g_V^p = \frac{1}{2} - 2 \sin^2 \theta_W$, $g_V^n = -\frac{1}{2}$. In (15), diagonal charge radii $\langle r_{\nu_\ell}^2 \rangle \equiv \langle r_{\nu_{\ell\ell}}^2 \rangle$ are separated out, which make a contribution by adding $\tilde{Q}_{\ell\ell}$ to the weak vector coupling constant g_V^p , which is equivalent to the above-mentioned shift (281).

Limits both for the diagonal neutrino effective charge radii $\langle r_{\nu_e}^2 \rangle$ and $\langle r_{\nu_\mu}^2 \rangle$ and for the transition charge radii $\langle r_{\nu_{e\mu}}^2 \rangle$, $\langle r_{\nu_{e\tau}}^2 \rangle$, and $\langle r_{\nu_{\mu\tau}}^2 \rangle$ were obtained in [59] by thorough fitting of the COHERENT data (by considering time dependence of the data, uncertainties of the neutron distribution in the ^{133}Cs and ^{127}I nuclei, etc.). For example, the limit obtained for the muon neutrino charge radius is

$$-8 \times 10^{-32} < \langle r_{\nu_\mu}^2 \rangle < 11 \times 10^{-32} \text{ cm}^2.$$

Limits (90% CL) on the transition radii were obtained for the first time in the form

$$\left(\left| \langle r_{\nu_{e\mu}}^2 \rangle \right|, \left| \langle r_{\nu_{e\tau}}^2 \rangle \right|, \left| \langle r_{\nu_{\mu\tau}}^2 \rangle \right| \right) < (22, 38, 27) \times 10^{-32} \text{ cm}^2.$$

This information is of interest for verifying a number of New Physics models which involve mechanisms for generation of the neutrino effective transition charge radii [102]. Similar investigations were carried out in [101].

Light sterile neutrinos were introduced as a possible explanation of short-baseline neutrino oscillations at $\Delta m^2 \sim 1 \text{ eV}^2$. In [103], an experiment was proposed for the direct search for active-to-sterile neutrino oscillations based on coherent elastic neutrino-nucleus scattering. As is known, the character of neutrino oscillations depends mainly on the neutrino energy and the distance the neutrino has traveled. Since energies of initial neutrinos cannot be reconstructed as a result of CEvNS, it was proposed to

detect the above oscillations (in the active neutrino disappearance mode) by explicitly using the dependence on the distance, which is well determined by the location of the target relative to the accelerator-based neutrino source (e.g., SNS). According to [103], this experiment could ultimately prove existence of sterile neutrinos and set limits on their mixing parameters.

The search for sterile neutrinos in the above mass region by measuring the counting rate of CEvNS events from a pulsed neutrino source (of the SNS type) was also proposed in [62]. Since CEvNS is a unique process for all active types of neutrino, this is a possible way to measure the total neutrino flux in the framework of the Standard Model. Using measurements at different baselines and taking into account timing for event reconstruction, one can considerably reduce systematic uncertainties and independently measure fluxes of ν_μ , ν_e , or $\bar{\nu}_\mu$ neutrinos. In [62], it was shown that the 100-kg CsI detector will be sensitive to a considerable region of the parameter space of a light sterile neutrino.

Capabilities of contemporary and future CEvNS experiments in researching electromagnetic neutrino interactions were studied, for example, in [104]. Here are a few useful relations from that work. In the simplest $SU(2)_L \otimes U(1)_Y$ model, the Majorana transition (from mass state i to mass state $j \neq i$) the neutrino magnetic moment has the form [99, 105]

$$\begin{aligned} \mu_{ij}^M &= -\frac{3ieG_F}{16\pi^2\sqrt{2}} (m_{\nu_i} + m_{\nu_j}) \\ &\times \sum_{\alpha=e,\mu,\tau} \text{Im} \left[U_{\alpha i}^* U_{\alpha j} \left(\frac{m_{l\alpha}}{M_W} \right)^2 \right], \end{aligned} \quad (149)$$

where G_F is the Fermi constant, m_{ν_i} is the mass of the neutrino state ν_i , $U_{\alpha i}$ is the mixing matrix element, and $m_{l\alpha}$ and M_W are the masses of the charged lepton and the W boson. The effective neutrino magnetic moment $(\mu_\nu^F)^2$, which can in principle be experimentally observable, is expressed in terms of the matrix of the neutrino magnetic moments and the amplitude of positive and negative neutrino helicity states [104].

For reactor-based experiments (with antineutrinos) there is the effective Majorana transition moment (in the neutrino flavor basis) [106] denoted in [104] by $(\mu_{\nu,\text{reactor}}^F)^2$. Generally speaking, it differs from a similar parameter measured in the accelerator-based setup because in the latter case it is possible to distinguish between the so-called prompt and delayed neutrino fluxes [59, 107]. In [104], it is proposed to use both prompt $(\mu_{\nu,\text{prompt}}^F)^2$ and delayed $(\mu_{\nu,\text{delayed}}^F)^2$ effective transition moments of Majorana neutrinos.

The idea of the analysis was based on that the contribution from nonstandard electromagnetic neutrino

interactions in the CEvNS experiment with a quite low recoil energy threshold (below keV) can be not only noticeable but even dominant [108], which, in turn, should lead to distortion of the nuclear recoil spectrum expected in the Standard Model. Since the standard weak interaction retains helicity while the electromagnetic interaction proceeds with helicity violation, their contributions can be added at the level of cross sections. Therefore, the data analysis [104] was performed using the sum of cross sections

$$\left(\frac{d\sigma}{dT_A}\right)_{\text{total}} = \left(\frac{d\sigma}{dT_A}\right)_{\text{SM}} + \left(\frac{d\sigma}{dT_A}\right)_{\text{EM}}.$$

For the differential CEvNS cross section for scattering of neutrinos with energy E_ν by the nucleus (A, Z) , the “standard” formula was used [1, 35, 39]

$$\begin{aligned} \left(\frac{d\sigma}{dT_A}\right)_{\text{SM}} &= \frac{G_F^2 m_A^2}{\pi} \\ &\times \left[G_V^2 \left(1 - \frac{m_A T_A}{2E_\nu^2}\right) + G_A^2 \left(1 + \frac{m_A T_A}{2E_\nu^2}\right) \right] F^2(q^2), \end{aligned} \quad (150)$$

where T_A is the nuclear recoil kinetic energy, and m_A is the mass of the target nucleus with Z protons and $N=A-Z$ neutrons. The vector G_V and axial-vector G_A contributions [109] were given by the formulas

$$\begin{aligned} G_V &= \left[2(g_u^L + g_u^R) + (g_d^L + g_d^R) \right] Z \\ &\quad + \left[(g_u^L + g_u^R) + 2(g_d^L + g_d^R) \right] N, \\ G_A &= \left[2(g_u^L - g_u^R) + (g_d^L - g_d^R) \right] (\delta Z) \\ &\quad + \left[(g_u^L - g_u^R) + 2(g_d^L - g_d^R) \right] (\delta N), \end{aligned}$$

where $(\delta Z) = Z_+ - Z_-$, $(\delta N) = N_+ - N_-$, $Z_+(N_+)$, and $Z_-(N_-)$ are the total numbers of protons (neutrons) with spins up and spins down [7]. In the analysis [104], the symmetrized Fermi parametrization (141) [82] was used for the nuclear form factor $F(q^2)$, and the left and right coupling constants of the u and d quarks with the Z boson were taken by considering radiative corrections [110]. The electromagnetic cross section was parametrized in [104] using the expression

$$\left(\frac{d\sigma}{dT_A}\right)_{\text{EM}} = \frac{\pi a_{\text{EM}}^2 \mu_\nu^2 Z^2}{m_e^2} \left(\frac{1 - T_A/E_\nu}{T_A} \right) F^2(q^2). \quad (151)$$

On the basis of the careful statistical analysis of the available data using formula (151) and expression (285), limits on the above-mentioned effective Majorana transition moments and on more fundamental parameters of theory were obtained. Particular details can be found in [104]. It was shown that future CEvNS experiments with low-threshold detectors could noticeably improve the current limits on the components of the transition Majorana neutrino magnetic moment obtained from the Borexino data [104].

The role of CEvNS in obtaining more accurate limits on the neutrino magnetic moment was also discussed in [101, 111].

An interesting possibility of using coherent bremsstrahlung on the nucleus $\nu A \rightarrow A\nu + \gamma$ to estimate the neutrino mass by experimentally determining the position of the photon spectrum end-point energy was proposed in [60]. Unfortunately, as the authors showed, the required high power of the monochromatic neutrino source and/or required large detector mass do not allow this method to compete with traditional electron spectrum endpoint measurements in nuclear beta decay. Interestingly, according to [60], the process $\nu A \rightarrow A\nu + \gamma$ is sensitive to the nature of the neutrino, i.e., it is in principle different for the Dirac and Majorana neutrinos. However, the fundamental difference of the photon spectra for these cases is on the order of m_ν/E_ν , which is extremely small despite the fact that the neutrino in the final state turns out to be almost at rest. Thus, the “theorem of Dirac–Majorana indistinguishability” remains unchanged, even if the indistinguishability condition in the form of $E_\nu \gg m_\nu$ is applicable to the initial state and may not be fulfilled in the final state ($E_\nu \approx m_\nu \approx 0$) [60].

The Weinberg angle $\sin^2 \theta_W$ is one of the main parameters of the Standard Model measured with a high accuracy at the Z pole [110]. This important quantity was also measured at very low transferred momenta, though with a noticeably lower accuracy [112]. The main results in this energy range were obtained from measurements of weak nuclear charges and atomic parity violation. The low-energy coherent elastic neutrino–nucleus scattering provides new independent information for measurement of the weak mixing angle in this region. In [113], it was shown that CEvNS experiments have a noticeable potential for achieving a high accuracy in obtaining a “low-energy” value of $\sin^2 \theta_W$ provided that systematic uncertainties are reduced. A similar investigation was carried out in [65, 114].

The influence of the COHERENT data on the accuracy of the determination of $\sin^2 \theta_W$ was discussed in [56], where, on the basis of these data and the data on atomic parity violation (APV) on cesium nuclei, a new result $\sin^2 \theta_W = 0.239_{-0.007}^{+0.006}$ is given for low transferred momentum processes, which is quite comparable with the Standard Model prediction $\sin^2 \theta_W = 0.23857(5)$.

8.1.5. New Physics or neutrino nonstandard interactions and CEvNS. The question of neutrino nonstandard interactions (NSIs) have long and widely been discussed in the literature. Continuous activity in global analysis of neutrino (mainly oscillation) data for obtaining constraints on NSIs can be exemplified by [65, 101, 114, 115].

Actually, NSIs are a manifestation of the so-called New Physics, i.e., particles and interactions that do not fall within the Standard Model, in the neutrino (more generally, neutral lepton) sector. Observation of CEvNS in the COHERENT experiment stimulated greater interest in NSIs since it actually gave rise to an entirely new (though expected) source of information on neutrinos and their interaction with matter. A rather precise measurement of the CEvNS spectrum within a reasonable time seems to be quite real. Obviously, existence of any new fundamental interactions of neutrinos with matter may increase, decrease, or noticeably distort the spectrum shape and the detection frequency of CEvNS events.

In [116], mutual complementarity of accelerator- and reactor-based CEvNS experiments was investigated from the point of view of obtaining significant constraints on nonstandard neutrino interactions. In particular, it was shown that simultaneous analysis of the results of these experiments, each with at least two different target materials, may appreciably improve the accuracy of the analysis and remove degeneracy between flavor-diagonal neutrino NSIs, which are hard to constrain in neutrino oscillation experiments. For example, considering four types of these parameters, the authors stated that they all can be measured with a rather high accuracy in next-generation experiments [116].

In [117], the authors investigated existing constraints and future sensitivity of experiments on coherent elastic neutrino–nucleus scattering to various Standard Model extensions through introduction of an additional Z' boson (additional neutral currents). The effect of the Z' boson boils down to redefinition of the neutrino–quark coupling constants g_V and g_A by taking into account the mass of the Z' boson and its mixing angle with the Standard Model Z boson. The results of the neutrino experiments on a fixed target, with atomic parity violation, with solar neutrinos, etc. were analyzed. The authors demonstrated a significant dependence of these constraints on nonuniversality of the Z' boson coupling with Standard Model particles and the importance of CEvNS for this kind of analysis [117].

A similar analysis, but for small (below 50 MeV) and large masses of the new intermediate Z' boson was performed in [118]. The results turned out to be rather sensitive to the mass of this boson, and at large masses degeneracy was observed between the New Physics parameters, which greatly diminished the significance of the limits obtained [118].

As an example of NSI estimation from the COHERENT data, we will give a slightly more detailed consideration to [119]. To analyze CEvNS processes for the presence of New Physics, one commonly uses neutrino nonstandard interactions, which

can be parametrized in the form of the effective four-fermion interaction Lagrangian

$$\mathcal{L}_{\text{eff}}^{\text{NSI}} = -\sqrt{2}G_F \sum_{q=u,d} \bar{\nu}_i \gamma^\mu (1 - \gamma_5) \nu_j \bar{q} (\epsilon_{ij}^{qV} + \gamma_5 \epsilon_{ij}^{qA}) q. \quad (152)$$

Here $\epsilon_{ij}^{q(V,A)}$ are the free parameters that are usually already limited to an extent by the neutrino scattering and neutrino oscillation data, $q = u, d$ (see, for example, [120]). These parameters specify the strength of new neutrino interactions with respect to G_F . In the maximum general form, NSIs can be defined as a set of all Lorentz-invariant interactions of neutrinos with first-generation quarks without derivatives (see, for example, [121])

$$\mathcal{L}_{\text{eff}}^{\text{NGI}} = \frac{G_F}{\sqrt{2}} \sum_X \bar{\nu} \Gamma^X \nu \bar{q} \Gamma_X (C_X^q + i\gamma_5 D_X^q) q. \quad (153)$$

Here $\Gamma^X = \{1, i\gamma^5, \gamma^\mu, \gamma^\mu \gamma^5, \sigma^{\mu\nu}\}$ is the total set of 4×4 matrices, where $\sigma^{\mu\nu} = i[\gamma^\mu, \gamma^\nu]/2$.

A more or less standard procedure of going from the neutrino–quark interaction Lagrangian to the effective neutrino–nucleus interaction begins with going from quarks to nucleons by averaging quark currents over nucleon states and then going from nucleons to the nucleus by averaging nucleon currents over nuclear states and introducing the corresponding nuclear form factors (for more detail, see, for example, [119, 122]). As a result, the neutrino–nucleus effective nonstandard interaction Lagrangian from [119] takes the form

$$\begin{aligned} \mathcal{L}_{\nu N} \sim & \sum_{X=S,V,T} \bar{\nu} \Gamma_X \nu \bar{N} C_X \Gamma_X N \\ & + \sum_{(X,Y)=(P,S),(A,V)} \bar{\nu} \Gamma_X \nu \bar{N} i D_X \Gamma_Y N, \end{aligned} \quad (154)$$

where the coefficients C_X and D_X parametrizing the effective νN interaction are defined by the (quark) parameters C_X^q and D_X^q from (153) and take the form

$$\begin{aligned} C_S &= Z \sum_{q=u,d} C_S^q \frac{m_p}{m_q} f_{T_q}^p + (A - Z) \sum_{q=u,d} C_S^q \frac{m_n}{m_q} f_{T_q}^n, \\ C_V &= Z(2C_V^u + C_V^d) + (A - Z)(C_V^u + 2C_V^d), \\ C_T &= Z(\delta_u^p C_T^u + \delta_d^p C_T^d) + (A - Z)(\delta_u^n C_T^u + \delta_d^n C_T^d). \end{aligned} \quad (155)$$

Numerical values of the structure parameters from the last line are [119] $\delta_u^p = 0.54$, $\delta_d^p = -0.23$, $\delta_u^n = -0.23$, and $\delta_d^n = 0.54$. Expressions for the coefficients D_p are obtained from C_S by the replacement of C_S^q with D_p^q , while for D_A they are obtained from C_V by the replacement of C_V^q with D_A^q [119]. Relations (155) connect effective parameters of the νN interaction with constants of fundamental interaction Lagrangian (153). Then, the authors [119] use these parameters in the

expression for the differential cross section of CEvNS by the target nucleus of the type a in the form

$$\begin{aligned} \frac{d\sigma^a(q^2)}{dE_r} &= \frac{G_F^2}{4\pi} m_{N_a} N_a^2 F(q^2) \\ &\times \left[\frac{\xi_S^2 E_r}{E_r^{\max}} + \xi_V^2 \left(1 - \frac{E_r}{E_r^{\max}} - \frac{E_r}{E_V} \right) \right. \\ &\left. + \xi_T^2 \left(1 - \frac{E_r}{2E_r^{\max}} - \frac{E_r}{E_V} \right) - \frac{RE_r}{E_V} \right], \end{aligned} \quad (156)$$

where $E_r^{\max} \approx \frac{2E_V^2}{m_{N_a}}$ is the maximum nuclear recoil energy, and the expressions

$$\begin{aligned} \xi_S^2 &= \frac{C_S^2 + D_P^2}{N^2}, \quad \xi_V^2 = \frac{C_V^2 + D_A^2}{N^2}, \\ \xi_T^2 &= 8 \frac{C_T^2}{N^2}, \quad R = 2 \frac{C_S C_T}{N^2} \end{aligned}$$

also inexplicitly depend on the target nucleus type a . Formula (156) is a simplified expression (4.9) from [121]. It is this expression for the CEvNS cross section (under various simplifying assumptions about relative significance of various parameters) that the authors [119] compared with the COHERENT data and obtained constraints on all significant parameters both in the one-parameter fit and in simultaneous fitting of the data with two parameters. The results were presented in the form of a set of correlation plots, e.g., in the parameter planes $C_S^u - C_V^d$, $C_S^u - C_T^d$, $C_V^u - C_T^d$, etc. [119].

Concluding their multifaceted and careful analysis, the authors stated that the results were the most rigorous estimates for effective generalized neutrino–quark interactions when masses of the carriers of these interactions are on the order of 1 GeV/ c^2 [119].

On this occasion, we note that, in our opinion, in [119], where approximate formulas from [121] are used, the question of difference between the coherent elastic νA and $\bar{\nu} A$ scatterings was tackled explicitly for the first time. The main conclusion that this difference can be ignored under reasonable assumptions on, for example, smallness of the axial-vector contribution or absence of simultaneous contributions from scalar and tensor NSIs appears to be quite justified and agrees with the exact calculations and conclusions from [20] or formulas (132) and (133).

However, justification of these approximate formulas in [121] is rather doubtful though the formulas themselves for the spin-zero nucleus look quite traditional

$$\begin{aligned} \frac{d\sigma}{dT} &= \frac{\sigma_{s=0}^{\text{SM}}}{M} \left(1 - \frac{T}{T_{\max}} \right), \\ \sigma_{s=0}^{\text{SM}} &\equiv \frac{G_F^2 [N - (1 - 4s_W^2)Z]^2 F^2(q^2) M^2}{4\pi}. \end{aligned} \quad (157)$$

The main problem is that the authors [121] simply declare at the very beginning that they calculate cross sections for the coherent νA scattering (and/or $\bar{\nu} A$ scattering) without considering in any way the coherence condition for the nucleus (that its quantum state does not change).

This is actually reduced to the fact that their matrix element $i\mathcal{M}^{ss'}(\nu A \rightarrow \nu A)$ looks like the matrix element of the (anti)neutrino scattering off a structureless scalar (since $J = 0$) particle multiplied by the weak nuclear charge $Q_W = N - (1 - 4s_W^2)Z$ and the form factor $F(q^2)$, namely

$$\begin{aligned} i\mathcal{M}^{ss'}(\nu A \rightarrow \nu A) &= i \frac{\sqrt{2}}{2} G_F Q_W F(q^2) \\ &\times g_L^{\nu}(p_2 + k_2)^{\mu} \bar{\nu}^s(p_1) \gamma^{\mu} (1 - \gamma^5) \nu^{s'}(k_1), \end{aligned}$$

where s and s' are the (anti)neutrino helicity. Then, the square of the matrix element is calculated by the standard method

$$\begin{aligned} |i\mathcal{M}|^2 &= \sum_{ss'} |i\mathcal{M}^{ss'}|^2 32 G_F^2 Q_W^2 \\ &\times F^2(q^2) M_A^2 E_V^2 \left(1 - \frac{T_A}{E_V} - \frac{M_A T_A}{2E_V^2} \right), \end{aligned} \quad (158)$$

where, as usual, M_A is the mass of the target nucleus, E_V is the neutrino energy, and T_A is the kinetic energy of the recoil nucleus. After slight simplifications Eq. (157) can be obtained from (158).

To be fair, it should be mentioned that the authors [121] discuss the case where the nucleus has the non-zero spin and the difference of the νA scattering from the $\bar{\nu} A$ scattering. They believe that the nonzero nuclear spin, $J \neq 0$, quite insignificantly affects the formula for the cross section when $J = 0$. Their explanation is that the (heavy) nucleus contains many spin 1/2 fermions (protons and neutrons), which almost compensate for each other's spins while making up the nucleus. If one proton flips its spin, the nuclear spin can change, e.g., from 0 to 1. Since the authors believe that the coherent νA scattering is insensitive to the state of one proton in the nucleus, they draw a conclusion (“suspect” in the original text) that there should be no significant difference between zero and nonzero spins as long as the nuclear spin itself is not very high.

Following this logic, the authors [121] give the matrix elements of the coherent νA and $\bar{\nu} A$ scattering off the nucleus (as a point particle) with $J = 1/2$ in the form

$$\begin{aligned} i\mathcal{M}^{r'ss'}(\nu A \rightarrow \nu A) &= i \frac{G_F}{\sqrt{2}} Q_W F(q^2) g_L^{\nu} \\ &\times [\bar{u}^{r'}(k_2) \gamma^{\mu} u^r(p_2)] [\bar{u}^{s'}(k_1) \gamma^{\mu} (1 - \gamma^5) u^s(p_1)], \end{aligned} \quad (159)$$

$$i\mathcal{M}^{r'rss'}(\bar{\nu}A \rightarrow \bar{\nu}A) = i\frac{G_F}{\sqrt{2}}Q_W F(q^2)g_L^\nu \times [\bar{u}^{r'}(k_2)\gamma^\mu u^r(p_2)] [\bar{\nu}^s(p_1)\gamma^\mu (1 - \gamma^5)\nu^{s'}(k_1)]. \quad (160)$$

Here $\bar{u}^{r'}(k_2)$ and $u^r(p_2)$ are the final and initial Dirac spinors of the point nucleus with spin $1/2$. Calculating the Dirac matrix traces in a standard way, the authors [121] obtain the squares of matrix elements (159) and (160) in an entirely identical form

$$|\mathcal{M}|^2 = \sum_{ss'} \frac{1}{2} \sum_{rr'} |i\mathcal{M}^{r'rss'}|^2 = 32G_F^2 Q_W^2 F^2 (g_L^\nu)^2 \times M_A^2 E_\nu^2 \left(1 - \frac{T_A}{E_\nu} - \frac{M_A T_A}{2E_\nu^2} + \frac{T_A^2}{2E_\nu^2}\right). \quad (161)$$

Then, for the cross sections there is also the same expression

$$\left. \frac{d\sigma}{dT_A} \right|_{J=1/2} = \frac{G_F^2 (2g_L^\nu Q_W)^2 F^2(q^2)}{4\pi} \times M_A \left(1 - \frac{T_A}{E_\nu} - \frac{M_A T_A}{2E_\nu^2} + \frac{T_A^2}{2E_\nu^2}\right). \quad (162)$$

According to [121], this coincidence of the νA and $\bar{\nu}A$ cross sections arises from the assumption of parity conservation in the interaction of the nucleus with the Z boson or from taking into consideration the vector current alone. Obviously, consideration of the axial-vector current, i.e., $\gamma^\mu \gamma^5$ between $\bar{u}^{r'}(k_2)$ and $u^r(p_2)$, leads to the difference of the νA and $\bar{\nu}A$ cross sections, which the authors [121] neglect.

Cross section (162) differs from cross section (157) for the nucleus with $J = 0$ by “small negligible corrections”, as the authors write. For this reason, the authors [121] considered only spin-zero nuclei with the vector character of weak interaction.

In this connection, we would specially mention that the summation in (161) is over all possible projections of the nuclear spin, which is unacceptable when the nucleus remains in its initial state, that is, interacts coherently. The same applies to the derivation of formula (156) in [121] for the case of exotic neutrino interactions given in the form of Lagrangian (153). Also, the modulus square of the νA scattering amplitude (formula (4.7) from [121]), which follows from the expression like (154), can be obtained in a standard way by summing over final (r') and averaging over initial (r) projections of the nuclear spin $|\mathcal{M}|^2 = \sum_{ss'} \frac{1}{2} \sum_{rr'} |\mathcal{M}^{s'sr'r}|^2$ (formula (4.8) from [121]), from which formula (156) follows.

In other words, justification of expressions for cross sections of coherent elastic (νA) $\bar{\nu}A$ scattering in [121] appears to be too simplified and inconvincible because, at a minimum, ignores unchangeability of

the nuclear state after interaction (coherence), as was done from first principles in [20]. Therefore, even the most thorough and careful analysis of the data (like the one in [119]) using formulas like (156) from [121] may well lead to incorrect results.

In [122], the potential of the COHERENT and CONUS experiments [123] was investigated from the point of view of obtaining constraints on another rather particular version of nonstandard neutrino interactions resulting from an additional light scalar particle ϕ interacting with both neutrinos and quarks. The authors considered two versions of this interaction, with conservation and violation of the lepton number. Both versions correspond to the spin-flip scattering and therefore make an additive contribution to the total cross section (no interference with the CEvNS amplitude). It is shown that current (future) experiments are sensitive to masses of these scalar particles at a level of several MeV and to coupling constants at the scale of 10^{-4} (10^{-6}). If masses of these scalar particles will be on the order of the incident neutrino energy, they could be observed through the characteristic distortion of the spectrum shape [122].

Unfortunately, the authors of this work, like the authors of [121], when deriving formulas for coherent νA scattering via exchange of the ϕ particle, also sum over all possible nuclear spin states. Here, coherence is again declared rather than constructively used in derivation of cross section formulas.

In [111], they also investigated the potential for observation of New Physics in the CEvNS process using modern low-temperature bolometers located near nuclear reactors. Contributions to CEvNS from the neutrino magnetic moment, nonstandard interactions, and simplified models involving the massive scalar or intermediate vector boson were considered. Targets of Ge, Zn, Si, CaWO_4 , and Al_2O_3 were examined. It is shown that on achieving the precise measurement of the CEvNS energy spectrum at a level of few percent up to the energy on the order of tens of eV, future experiments will be able to improve by two orders of magnitude the limits on both the neutrino magnetic moment and the masses of the intermediate particles mentioned [111].

The question of how the COHERENT data can help set constraints on or even eliminate so-called “large NSIs” was discussed in [124]. This sort of NSI can affect the interpretation of the data from oscillation experiments. In particular, the solution known as LMA-Dark, which requires $\theta_{12} > \pi/4$ and a certain type of NSI with the effective coupling on the order of G_F (hence the term “large NSIs”), is not in conflict with these data so far (and is even quite competitive with the standard description of neutrino oscillations). Contrary to popular expectations, there is a class of models that can lead to these large NSIs. The authors [124] state that these models can in principle be veri-

fied in CEvNS experiments, such as COHERENT and upcoming reactor-based experiment CONUS [123].

The effect of NSIs with CP violation on the CEvNS spectrum measurement in the COHERENT sodium–iodine, liquid argon, and germanium detectors was considered for the first time in [125]. The authors identified the region in the parameter space for which the event acquisition rate always includes a dip and another region for which this dip never occurs. It is shown that this dip can be used to restrict CP-violating effects in NSIs. In the dip-free region, the CP-violating parameters may well look like Standard Model CEvNS predictions [125].

Prospects for obtaining information beyond Standard Model physics by simultaneously analyzing the energy and time distributions of the CEvNS data from the COHERENT experiment were studied in [107]. Consideration of rather light intermediate additional bosons with masses ranging from 10 to 1000 MeV/ c^2 demonstrated a better description of the combined data if such bosons are included than within the Standard Model (difference at a level of 2σ). This synchronization allows additional information on the distribution of electron and muon neutrinos to be gained, which is impossible if analyzing the recoil nucleus energy distribution alone. Therefore, there arises an additional possibility of suppressing the background, obtaining more accurate data on the nuclear structure, and imposing harder constraints on New Physics [107].

Concluding the discussion of neutrino nonstandard interactions, it is worth touching on [61], where NSIs are subjected to complete formalization within the effective field theory (EFT). Contributions of NSIs to the cross sections for coherent interactions (and deep inelastic scattering) from interactions of neutrinos with light quarks, leptons, gluons, and photons were parametrized in the form of all possible dimension 5, 6, and 7 operators. Under a natural assumption of single operator dominance, the authors obtained constraints on the Wilson coefficient corresponding to the dominant operator (or weight coefficient of the operator) using the COHERENT and CHARM data. The high potential of future CEvNS experiments in constraining the parameter space of possible values of the considered effective operators was pointed out [61]. The authors really did a big job (60 pages) but, unfortunately, of little value, to our mind. The reason is a too simplified interpretation of coherent neutrino–nucleus scattering similar to the one mentioned above (see discussion of [121] and [119]).

For example, from the very beginning the authors [61] assert that neutrons and protons in a nucleus are nonrelativistic, and they are well described by the chiral EFT with nonrelativistic nucleons. It is assumed that the momentum q transferred in CEvNS is small,

and therefore the nucleus remains intact and neutrons and protons are nonrelativistic during the entire scattering. Next, since in the COHERENT experiment a typical q is in the region of 30–70 MeV, which is much below the cutoff parameter in the chiral EFT, $\Lambda \sim \mathcal{O}(1 \text{ GeV})$, effective neutrino interactions can be described within the chiral EFT [61]. As was specially discussed in section 3, this interpretation is at least incompatible with the coherence condition in the neutrino–nucleus scattering. Then, the response of the nucleus to NSIs is given in [61] by the expression

$$\frac{d\sigma_A}{dE_R} = \frac{m_A}{\pi E_N^2} \overline{\mathcal{M}}^2,$$

where E_R is the nuclear recoil energy, and the square of the average scattering amplitude has the form

$$\overline{\mathcal{M}}^2 = \frac{1}{2J_A + 1} \sum_{\text{spins}} |\mathcal{M}|^2 = \frac{4\pi}{2J_A + 1} \times \sum_{\tau, \tau'=0,1} (R_M^{\tau\tau'} W_M^{\tau\tau'} + R_{\Sigma^-}^{\tau\tau'} W_{\Sigma^-}^{\tau\tau'} + R_{\Sigma^+}^{\tau\tau'} W_{\Sigma^+}^{\tau\tau'}). \quad (163)$$

Here J_A is the spin of the target nucleus, and $W_{M, \Sigma^-, \Sigma^+}(q)$ are the nuclear response functions. Summation over isospin indices τ, τ' is carried out by considering kinematic factors in the form

$$\begin{aligned} R_M^{\tau\tau'} &= \text{Tr}(P_L \not{l}_{0,\tau} \not{l}_{0,\tau'}), \\ R_{\Sigma^-}^{\tau\tau'} &= \text{Tr}(P_L \not{l}_{5,\tau}^{j\dagger} \not{l}_{5,\tau'}^j \hat{q}^j \hat{q}^j), \\ R_{\Sigma^+}^{\tau\tau'} &= \text{Tr}(P_L \not{l}_{5,\tau}^{j\dagger} \not{l}_{5,\tau'}^j) (\delta^{jj} - \hat{q}^j \hat{q}^j), \end{aligned}$$

which accumulate with $\hat{q} \equiv \vec{q}/q$ into Dirac structures l_0 and $\vec{l}_5$ (see formulas (34)–(35) in [61]) entering into the effective nonrelativistic Lagrangian of neutrino–nucleus interaction

$$\mathcal{L}_{\text{NR}} = (\bar{\nu} l_0 P_L \nu) \bar{N} \nu N_\nu + 2(\bar{\nu} \vec{l}_5 P_L \nu) \bar{N}_\nu S_N N_\nu.$$

It is seen that the summation in Eq. (163) is done over all projections of the spin of the initial and final nucleus and also summation is done over projections of the initial and final isospin, which admits possible changes in the spin and isospin due to interaction. As was already mentioned, this procedure has nothing in common with the concept of coherence.

Concluding the brief review of physical problems for CEvNS experiments, it should be mentioned that the novelty potential of the problems is high indeed. New works on this subject are regularly published (see, for example, [126–136]).

Unfortunately, the analyses of the already available data and the data of projected experiments are not always thoroughly physically substantiated. At a minimum, there is no clear (analytical) justification of applicability of the formulas used in the precision analysis of coherent (anti)neutrino–nucleus scattering. Coherence (or, which is almost equivalent, pro-

portionality of the cross section to the square of the neutron number) is often taken as given a priori simply because the transferred momentum is comparable to the inverse of the target nucleus radius. Probably, it might well have been acceptable before publication of [19, 20], where the coherent and incoherent regimes of the neutrino–nucleus scattering were consistently mathematically considered from first principles.

8.2. *Some of CEvNS Experiments and Their Capabilities*

This section is a brief overview of the most important experiments on detection and study of CEvNS. In addition to the already discussed COHERENT experiment, which uses beams of “accelerator” (anti)neutrinos¹⁹ from the decay of stopped pions produced in a spallation neutron source (SNS), the experiments to be discussed below are mainly supposed to use antineutrino beams from nuclear reactors or natural neutrino sources, such as the Sun or a supernova. Modern experiments with reactor and accelerator (anti)neutrinos are reviewed, for example, in [137–140]. Interest in the CEvNS research was also demonstrated at the European Spallation Source [141].

The research program of the TEXONO collaboration is quite wide [142]. In particular, it involves search for and observation of CEvNS initiated by (anti)neutrinos from nuclear reactors. Following this goal, a nuclear recoil energy threshold of 100–200 eV was achieved in the prototype low-threshold highly pure germanium detector. The facility with this low threshold was also supposed to be used for performing direct search for cold dark matter in the region of small masses and for increasing sensitivity of the search for neutrino magnetic moments [143].

In 2012, a new (Russian) project RED (Russian Emission Detector), a two-phase emission detector with liquid xenon as a working body, was proposed [144]. Among the objectives of the project, one of the primary tasks was observation of neutral-current coherent neutrino–nucleus scattering. The expected signals and background were initially estimated for two possible sites of the detector, the Kalinin Nuclear Power Plant (Russia) and the SNS at Oak Ridge National Laboratory (United States) [144]. After the relevant simulations and R&D, both sites looked appropriate for conducting the experiment, but now the use of RED for studying CEvNS only with a nuclear reactor beam is under discussion [145]. Note that the team that proposed the RED experiment was largely successful in the COHERENT experiment.

In 2014, it was proposed to use the gaseous detector, a three-section low-background proportional

counter, for detecting reactor antineutrinos through their coherent scattering off nuclei [146]. The working substance was supposed to be argon and xenon. It was shown that pulse *waveform* separation of the signal allowed quite efficient suppression of the background from electromagnetic noise and microphone effects in the energy range of 20 to 100 eV, where the main manifestation of the CEvNS effect was expected. Based on the analysis of the neutron background generated by cosmic-ray muons, a scheme of a possible experimental facility was presented [146]. After 2014, no works related to this proposal have been published.

The idea of using CEvNS in combination with monoenergetic neutrino sources to search for New Physics, for example, through detection of light sterile neutrinos underlay the special investigation performed in [147] to find out whether it was possible to optimize cryogenic solid-state bolometers for achieving extremely low thresholds for nuclear recoil kinetic energy at a level of 10 eV. The sensitivity of an array of such detectors to oscillations of neutrinos to sterile states was investigated, and prospects for their use was shown [147].

In 2013, a possibility of searching for CEvNS using the far-off-axis ($>45^\circ$) Fermilab Booster neutrino beam with the energy below 50 MeV was studied, and the corresponding proposal was made. It was proposed to use the detector situated transverse to a high-energy neutrino beam production target. This detector could be sensitive to low-energy neutrinos resulting from pion decay at rest. The background and the sensitivity were investigated, and it was shown that a one-phase liquid-argon detector with one ton of working substance might detect (at the level of 7.5σ) CEvNS at the nuclear recoil energy threshold of 25 keV [148].

In the Coherent Neutrino–Nucleus Interaction Experiment (CONNIE), low-noise CCDs are used to detect low recoil energies of silicon nuclei caused by reactor antineutrinos [149]. It is expected that these devices will allow a record nuclear recoil energy threshold of 28 eV [150]. Despite the known COHERENT achievements, CEvNS has not been observed at recoil energies below a few keV yet. In 2016, the first results were obtained from a set of detectors with the active mass of 73.2 g (12 CCDs), which operated at a distance of 30 m from the core of the nuclear reactor with the thermal power of 3.8 GW. Neutrino events were searched for by comparing the data collected with the reactor on (2.1 kg day) and off (1.6 kg day). No data excess was observed with the reactor on. Nevertheless, a record sensitivity level was achieved up to the recoil energies of about 1 keV (0.1 keV electron-equivalent) [151].

At the end of 2016, the RICOCHET experiment was proposed to search for CEvNS using low-temperature bolometers at the Chooz reactor complex (France) as a source of reactor (anti)neutrinos [152]. Potential CEvNS sensitivity was investigated for this experiment, which is a composite detector (bolomet-

¹⁹In the COHERENT experiment, target nuclei have a nonzero spin, and therefore, according to formulas (128) and (129), antineutrino-induced CEvNS differs from neutrino-induced CEvNS.

ric array) of small germanium and zinc modules with the total mass of about 10 kg. It was shown that if correlation with the reactor power and the energy threshold of 100 eV are used for background–signal separation, a CEvNS signal could be detected within one year of running [152].

A small-scale ν -cleus (or NU-CLEUS) experiment was aimed at detecting CEvNS by detecting nuclear recoil energies down to 10 eV [153]. The detector consisted of low-threshold CaWO_4 and Al_2O_3 calorimeter arrays with a total mass of about 10 g and several cryogenic veto detectors operated at millikelvin temperatures. Segmentation of the detector ensured active identification of the backgrounds γ quanta, neutrons, and surface effects. The first prototype sapphire (Al_2O_3) device operated above ground (without shielding) achieved the energy threshold of 20 eV. A study of sensitivity to CEvNS in antineutrino beams from nuclear power plants showed that CEvNS discovery (5σ) might be possible within two weeks of measuring. It was demonstrated that real-time monitoring of nuclear power plants was feasible with this technology [153]. The use of large-volume cryogenic detectors to search for CEvNS and detect supernova neutrinos was earlier discussed in [154].

The sensitivity of bubble chambers like PICO-500 to supernova neutrinos through CEvNS was studied in [155]. The main problem is to ensure sensitivity of large-scale detectors to low recoil energies of about 10 keV arising from coherent elastic scattering of supernova neutrinos by target nuclei. Large superheated-fluid detectors like PICO-500, a bubble chamber filled with an active target containing 1 t of C_3F_8 , should see multiple-bubble events from CEvNS if the detector is active during a neutrino burst from a nearby (10 kpc) supernova. A possibility of using these bubble chambers as an independent method for studying supernovas similar to SN 1987A with particular sensitivity to the heavy-lepton channel was discussed [155].

At the Nuclear Science Center at Texas University the Mitchell Institute Neutrino Experiment at Reactor (MINER) is conducted to search for CEvNS in close proximity (about 2 m) of the 1 MW TRIGA nuclear reactor core using low-threshold cryogenic germanium and silicon detectors. Simulations of experimental conditions performed in 2016 by considering coherent enhancement of the expected cross section, neutron, muon, and gamma backgrounds, and proposed proximity to the reactor estimated the expected number of signal events at 5–20 events/kg/day, while measurable nuclear recoil energies were supposed to fall within the range of 10 to 1000 eV [156]. We failed to find more recent information about this experiment.

As was mentioned above, observation of galactic neutrinos from supernovas is one of the main objectives of neutrino physics and low-energy astrophysics. In [76, 77], it was proposed to use a small-sized gaseous spherical TPC detector filled with a noble gas.

The idea of neutrino detection rested on the CEvNS effect. Estimations show that for a sphere with the radius of 3 m filled with xenon at 10 atm, with the recoil energy threshold of 100 eV, one can expect 300 to 500 events (depending on neutrino mass hierarchy). With the pressure raised to 50 atm, this could go up to 1500–2500 events. With ^{40}Ar , 150–200 events could be expected under the same conditions.

The second idea of these authors was to install a few of this easy-to-maintain and relatively inexpensive detectors (filled with argon at a relatively low pressure) at different places over the globe. This network of gaseous TPC detectors operating in coincidence with the energy threshold below 1 keV was proposed to use for monitoring outer space in the low-energy neutrino channel. While awaiting a signal from a supernova, these detectors can be easily operated (even by students) for a long time [76, 77].

A dual-phase gaseous (argon-based) TPC detector capable of measuring the nuclear recoil energy below 1 keV and detecting nuclear reactor antineutrinos through CEvNS was also proposed in [157]. This detector was supposed to be first used for measuring an important parameter—the nuclear ionization quenching factor in the yet unstudied recoil energy range of few keV and below [157].

In 2016, a team from the Max Planck Institute (Germany) proposed a CONUS (COherent NeUtrino Scattering) reactor experiment aimed at detecting CEvNS in the fully (as the authors state) coherent regime using low-threshold (240 eV) highly pure Ge detectors [158]. In 2017, an assembly of four modules (1 kg each) started data taking at a commercial nuclear power plant in Brockdorf (Germany) at a distance of 17 m from the reactor core (expected antineutrino flux over $10^{13}/\text{s}/\text{cm}^2$). For the experiment to be a success, it was necessary to well understand the neutron background conditions, because signals of nuclear recoil from neutron impacts fully imitate the desired CEvNS interactions. Particularly unpleasant for CONUS are events induced by neutrons accompanying thermal energy production.

In 2019, results of measuring background conditions at the detector site were presented for the first time. The presence of a highly thermalized neutron field correlated with thermal power of the reactor was shown. With appropriate shielding of the CONUS facility, the flux of these neutrons resulting from nuclear fission in the core was suppressed, according to the authors [123], by approximately a factor of 10^{20} . The γ -background was also studied using a highly pure Ge detector. A careful simulation by considering these data showed that the expected CEvNS signal was at least an order of magnitude higher than the reactor-induced background in the measurement region [123].

In [159], on the basis of the obvious requirement that the precise knowledge of the neutrino flux and energy spectrum is necessary for the search of New

Physics, an interesting possibility was studied of conducting a CEvNS experiment based on a ^{51}Cr source whose neutrino spectrum is known and activity can be monitored by a thermal sensor at a level of a few per mil. It was shown that with a 5 mCi source placed at a distance of 25 cm from the detector, under an exposure of about 55 days, one can expect 3900 (900) counts on a germanium (sapphire) target 2 dm^3 in volume with an energy threshold of 8 (20) eV [159].

It is worth mentioning that the EDELWEISS collaboration [160] achieved a record energy threshold of 60 eV for the 33.4 g cryogenic Ge detector in their direct search for light (masses below $1\text{ GeV}/c^2$) dark matter particles. This threshold was obtained through measurement of the nuclear recoil energy deposited in the bolometer using a dedicated (neutron-transmutation-doped) Ge thermal sensor with the thermal energy resolution of 17.7 eV [161]. With this, in addition to the most stringent constraints on spin-independent interactions of dark matter particles above $600\text{ MeV}/c^2$ and their spin-dependent interactions with protons and neutrons below $1.3\text{ GeV}/c^2$, the EDELWEISS experiment demonstrated unique capabilities of modern germanium bolometers in decreasing the energy threshold and consequently in being successfully used for searching and detecting CEvNS in the most promising region of “maximum low” nuclear recoil energies, where the coherence condition is most precisely fulfilled. This experience is highly important for the vGEN experiment [52] being conducted by the Russian team of the EDELWEISS collaboration at the Kalinin Power Plant.

The vGeN experiment is a new experiment at the Kalinin Power Plant aimed at observing coherent elastic neutrino scattering off germanium nuclei. It uses low-threshold HPGe detectors developed at JINR (Dubna, Russia) in cooperation with BSI (Baltic Scientific Instruments, Latvia). Electron antineutrinos from one of the Kalinin Power Plant units are used as a powerful neutrino source. It is supposed to observe CEvNS using the differential method, which compares spectra measured during the reactor operation and shutdown periods and spectra measured at different distances from the core of the operating reactor. For the setup with four HPGe detectors (400 g each) located at a distance of 10 m from the reactor core center, with an energy threshold of 350 eV, up to ten CEvNS events per day are expected. The sensitivity of the setup will be increased using new detectors with a total mass of up to 5 kg [52]. In 2018, a threshold of 200 eV was achieved in this project [50].

In conclusion, it is worth mentioning one of the first proposals (1983) to create [43] a real neutrino observatory for detecting MeV neutrinos through their elastic coherent scattering off nuclei and identifying recoil energy with a rather compact detector. This detector was supposed to be implemented in terms of the superconducting grain idea, which appeared to be

feasible through extrapolation of the already known techniques. It could measure the neutrino energy spectrum and would be insensitive to neutrino oscillations because it would have detected all neutrino types. It was supposed to be used for detecting reactor, supernova, solar, and terrestrial neutrinos, as well as accelerator neutrinos from stopped pion decay [43].

In 1979, a coherent detector based on total reflection of low-energy neutrinos from a plane surface was also proposed. The underlying idea was that those neutrinos coherently interacted with the detector as a whole (single) object, and it was proposed to detect the total momentum that the detector acquired due to reflection of the neutrino beam [162]. In a recent work [27], this possibility was shown to be untenable.

9. CONCLUSIONS

Let us sum up the main points of the above discussion.

(1) Detailed consideration is given to all elements of the new theoretical approach developed in [19] for describing the neutrino–nucleus process $\nu A \rightarrow \nu A^{(*)}$, where the target nucleus A can go to an excited state $A^{(*)}$ while remaining intact. The approach relies on the microscopic description of the nucleus as a bound state of its constituent nucleons, which is based on the multiparticle nuclear wave function of the general form. Generalization of this approach to the antineutrino–nucleus interaction $\bar{\nu} A \rightarrow \bar{\nu} A^{(*)}$ is given in [20]. In particular, it is shown for the first time that under the kinematic conditions considered a rather good approximation is equality of the $\nu A \rightarrow \nu A^{(*)}$ and $\bar{\nu} A \rightarrow \bar{\nu} A^{(*)}$ cross sections.

(2) Consideration of all possible initial and final quantum states of the nucleus (completeness condition) in a rather general form (i.e., irrespective of details of the model of the nucleus) within this approach makes it possible to obtain a unified description of elastic (coherent) and inelastic (incoherent) neutrino and antineutrino scattering off nuclei.

(3) It is shown that elastic interaction preserving the initial quantum state of the nucleus leads to quadratic enhancement of the corresponding observed cross section in terms of the nucleon number. At the same time, the observed cross section (for all possible) inelastic processes where the quantum state of the nucleus changes has a linear dependence on the nucleon number. In Freedman’s terms, these two classes of processes are referred to as coherent and incoherent, respectively.

(4) It is shown for the first time that the behavior of the elastic (coherent) and inelastic (incoherent) scattering is determined by the factors $|F_{p/n}(\mathbf{q})|^2$ and $(1 - |F_{p/n}(\mathbf{q})|^2)$, respectively, where $F_{p/n}(\mathbf{q})$ is the pro-

ton/neutron form factor normalized to unity and averaged over the initial nuclear state. These form factors control smooth transition from the elastic (coherent) to the inelastic (incoherent) regime of (anti)neutrino–nucleus scattering. In the general case, both regimes make a contribution to the observables. Domination of the elastic (coherent) or inelastic (incoherent) term is dictated by the relation between the $\mathbf{q}$ -dependent factors $A_{p/n}^2 |F_{p/n}(\mathbf{q})|^2$ and $A_{p/n} (1 - |F_{p/n}(\mathbf{q})|^2)$, where $\mathbf{q}$ is the momentum transferred to the nucleus.

(5) The elastic and inelastic $\nu(\bar{\nu})A$ processes discussed, due to their common nature resulting from weak neutron currents, turn out to be experimentally indistinguishable when the only observable is the target nucleus recoil energy. Therefore, in the experiments aimed at studying coherent neutrino scattering (at rather high energies) through detection of the recoil energy alone there can be an incoherent background indistinguishable from the main signal if nuclear deexcitation photons cannot be detected.

In particular, for the ^{133}Cs nucleus and the neutrino energy of 30–50 MeV the incoherent cross section may amount to 15–20% of the coherent cross section. Therefore, while in the COHERENT experiment (with the ^{133}Cs nucleus) “incoherent” γ rays from nuclear deexcitation are not detected, in the data of this experiment there are CEvNS events (i.e., the desired events of coherent elastic neutrino–nucleus scattering) and an inelastic impurity of about 15–20%. The precision analysis of the data from this kind of experiment should take into consideration the above-mentioned incoherence impurity.

(6) On the other hand, this incoherent contribution can be directly measured (estimated) through detection of photons emitted by target nuclei excited in inelastic processes. These photons should correlate in time with the neutrino beam and have an energy that is usually much higher than the nuclear recoil kinetic energy, which make them rather easy to detect; moreover, the number of these photons is governed by the inelastic-to-elastic cross section ratio.

(7) Simultaneous detection of two signals, the nuclear recoil energy and the energy of the nuclear deexcitation γ rays, allows the pure CEvNS contribution to be extracted, and thus the nuclear structure to be studied in detail and a precision search for New Physics to be carried out in this type of neutrino processes. Another way to perform precision study of CEvNS is based on the use of lower-energy (anti)neutrino beams and new-generation detectors with extremely low energy thresholds, which allows collecting and analyzing data in the region of very low recoil energies where inelastic processes are either strongly suppressed or simply impossible.

10. APPENDICES: DEFINITIONS, DERIVATION OF FORMULAS, AND NOTES

10.1. Construction of the n -Particle Wave Function of the Bound System (Nucleus) from Single-Particle States

10.1.1. Single-particle fermion quantum states. A fermion (spin $1/2$) state with mass m , a definite three-momentum $\mathbf{p}$, energy $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$, and projection of the spin on some direction s is defined in terms of the production operator $a_s^+(\mathbf{p})$ as [41]

$$|\mathbf{p}, s\rangle = \sqrt{2E_{\mathbf{p}}} a_s^+(\mathbf{p}) |0\rangle, \quad (164)$$

and there are the vacuum $\langle 0|0\rangle = 1$ and Lorentz-invariant²⁰ normalizations [41]

$$\langle \mathbf{k}, s | \mathbf{p}, r \rangle = (2\pi)^3 2E_{\mathbf{p}} \delta^3(\mathbf{p} - \mathbf{k}) \delta_{rs}. \quad (165)$$

The fermion state with a definite $\mathbf{x}$ can be given by the expression

$$\langle \mathbf{x} | = \langle 0 | \hat{\psi}(\mathbf{x}), \quad (166)$$

where $\hat{\psi}(\mathbf{x})$ is the field operator of the free particle in the Schrödinger representation (no time dependence) defined, for example, in the standard form [41], in terms of the production operators $a_r^+(\mathbf{p})$, $b_s^+(\mathbf{p})$ and annihilation operators $a_s(\mathbf{p})$, $b_r(\mathbf{p})$ as

$$\begin{aligned} \hat{\psi}(\mathbf{x}) &= \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{p}}{\sqrt{2E_{\mathbf{p}}}} \\ &\times \sum_{s=1,2} \left(e^{-i\mathbf{p}\mathbf{x}} a_s(\mathbf{p}) u(\mathbf{p}, s) + e^{i\mathbf{p}\mathbf{x}} b_s^+(\mathbf{p}) v(\mathbf{p}, s) \right), \\ \bar{\hat{\psi}}(\mathbf{x}) &= \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{p}}{\sqrt{2E_{\mathbf{p}}}} \\ &\times \sum_{r=1,2} \left(e^{-i\mathbf{p}\mathbf{x}} b_r(\mathbf{p}) \bar{v}(\mathbf{p}, r) + e^{i\mathbf{p}\mathbf{x}} a_r^+(\mathbf{p}) \bar{u}(\mathbf{p}, r) \right). \end{aligned} \quad (167)$$

The spinor functions $u(\mathbf{p}, s)$ and $v(\mathbf{p}, s)$ are solutions of the Dirac equation

$$(\gamma^\mu p_\mu - m)u(\mathbf{p}, s) = 0, \quad (\gamma^\mu p_\mu + m)v(\mathbf{p}, s) = 0, \quad (168)$$

and, considering the definition $\bar{u} \equiv [u^*]^T \gamma^0 = u^\dagger \gamma^0$, they are normalized as

$$\begin{aligned} \bar{u}(\mathbf{p}, s) u(\mathbf{p}, s') &= 2\delta_{ss'} m, \\ \bar{v}(\mathbf{p}, s) v(\mathbf{p}, s') &= -2\delta_{ss'} m, \quad \bar{u}(\mathbf{p}, s) v(\mathbf{p}, s') = 0. \end{aligned}$$

Solutions of equation (168) using Pauli matrix $\sigma = (\sigma^1, \sigma^2, \sigma^3)$ have the explicit form

²⁰Since $E_{\mathbf{p}} \delta^3(\mathbf{p} - \mathbf{k}) = E_{\mathbf{p}^*} \delta^3(\mathbf{p}^* - \mathbf{k}^*)$.

$$\begin{aligned} u(\mathbf{p}, s) &= \sqrt{E+m} \begin{pmatrix} \chi_s \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E+m} \chi_s \end{pmatrix} = \begin{pmatrix} \lambda_+ \chi_s \\ \lambda_- \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{|\mathbf{p}|} \chi_s \end{pmatrix}, \\ v(\mathbf{p}, s) &= \sqrt{E+m} \begin{pmatrix} \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E+m} \bar{\chi}_s \\ \bar{\chi}_s \end{pmatrix} = \begin{pmatrix} \lambda_- \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{|\mathbf{p}|} \bar{\chi}_s \\ \lambda_+ \bar{\chi}_s \end{pmatrix}, \end{aligned} \quad (169)$$

where $\lambda_{\pm} = \sqrt{E \pm m}$. The bispinors χ_s (and/or $\bar{\chi}_s$) can be chosen in the form of two orthogonal basis vectors

$$\begin{aligned} \chi_{\uparrow} &\equiv \chi_1 \equiv \chi_{+1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \text{and } \chi_{\downarrow} &\equiv \chi_2 \equiv \chi_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \end{aligned} \quad (170)$$

which are eigenvectors of the Pauli matrix $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ with eigenvalues +1 and -1 and are oriented along and against the 3rd (z) axis, respectively.

State (166) is the Dirac spinor and is normalized in accordance with the condition

$$\langle \mathbf{y} | \mathbf{x} \rangle = \delta^3(\mathbf{x} - \mathbf{y}) \hat{I}_{4 \times 4},$$

where the expression $\hat{I}_{4 \times 4}$ is a unit 4×4 matrix in the spin space.

Unit operators (operators “1”) in the single-particle basis can be given in the form

$$\hat{I}_{p,1} = \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{|\mathbf{p}, s\rangle \langle \mathbf{p}, s|}{2E_p}, \quad \hat{I}_{x,1} = \int d\mathbf{x} |\mathbf{x}\rangle \langle \mathbf{x}|. \quad (171)$$

The scalar product of the states defined by relations (164) and (166)

$$\langle \mathbf{x} | \mathbf{p}, s \rangle = u(\mathbf{p}, s) e^{i\mathbf{p} \cdot \mathbf{x}}, \quad (172)$$

allows expressing them in terms of each other

$$\begin{aligned} |\mathbf{p}, s\rangle &= \int d\mathbf{x} u^\dagger(\mathbf{p}, s) e^{-i\mathbf{p} \cdot \mathbf{x}} \langle \mathbf{x} |, \\ \langle \mathbf{x} | &= \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{u(\mathbf{p}, s) e^{i\mathbf{p} \cdot \mathbf{x}}}{2E_p} \langle \mathbf{p}, s|. \end{aligned} \quad (173)$$

It is evident that the expression for $\langle \mathbf{x} |$ in (173), which is given by (166), differs (by the normalization factor) from the nonrelativistic expression for the spin-zero state

$$\langle \mathbf{x} | = \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{e^{i\mathbf{p} \cdot \mathbf{x}}}{\sqrt{2E_p}} \langle \mathbf{p} |, \quad (174)$$

where $\langle \mathbf{p} |$ is given similarly to (164) but for the spin-zero particle.

The single-particle state $|\psi\rangle$ can be expressed in terms of the states $|\mathbf{p}, s\rangle$ and $|\mathbf{x}\rangle$ using the completeness condition in the form of unit operators (171)

$$\begin{aligned} |\psi\rangle &\equiv \hat{I}_{p,1} |\psi\rangle = \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{|\mathbf{p}, s\rangle \langle \mathbf{p}, s| \psi\rangle}{2E_p} \\ &= \int \frac{d\mathbf{p}}{(2\pi)^3} |\mathbf{p}, s\rangle \frac{\tilde{\psi}(\mathbf{p}, s)}{\sqrt{2E_p}}, \\ |\psi\rangle &\equiv \hat{I}_{x,1} |\psi\rangle \\ &= \int d\mathbf{x} |\mathbf{x}\rangle \langle \mathbf{x} | \psi\rangle \equiv \int d\mathbf{x} |\mathbf{x}\rangle \psi(\mathbf{x}). \end{aligned} \quad (175)$$

Here the designations $\tilde{\psi}(\mathbf{p}, s) \frac{\langle \mathbf{p}, s | \psi\rangle}{\sqrt{2E_p}}$ and $\psi(\mathbf{x}) = \langle \mathbf{x} | \psi\rangle$ are introduced.

From the conditions of normalization of states (175) in the form $\langle \psi | \psi \rangle = 1$, their own normalization conditions are obtained for the wave functions $\tilde{\psi}(\mathbf{p}, s)$ and $\psi(\mathbf{x})$

$$\begin{aligned} \langle \psi | \psi \rangle &= \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{\langle \psi | \mathbf{p}, s \rangle \langle \mathbf{p}, s | \psi \rangle}{2E_p} \\ &= \int \frac{d\mathbf{p}}{(2\pi)^3} |\tilde{\psi}(\mathbf{p}, s)|^2 = 1, \\ \langle \psi | \psi \rangle &= \int d\mathbf{x} \langle \psi | \mathbf{x} \rangle \langle \mathbf{x} | \psi \rangle \\ &= \int d\mathbf{x} |\psi(\mathbf{x})|^2 = 1. \end{aligned}$$

Or

$$\int \frac{d\mathbf{p}}{(2\pi)^3} |\tilde{\psi}(\mathbf{p}, s)|^2 = \int d\mathbf{x} |\psi(\mathbf{x})|^2 = 1. \quad (176)$$

The scalar function $\tilde{\psi}(\mathbf{p}, s)$ and the Dirac spinor $\psi(\mathbf{x})$ are related to each other by the Fourier transform²¹

$$\begin{aligned} \psi(\mathbf{x}) &= \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{\tilde{\psi}(\mathbf{p}, s)}{\sqrt{2E_p}} u(\mathbf{p}, s) e^{i\mathbf{p} \cdot \mathbf{x}}, \\ \tilde{\psi}(\mathbf{p}, s) &= \frac{u^\dagger(\mathbf{p}, s)}{\sqrt{2E_p}} \int d\mathbf{x} \psi(\mathbf{x}) e^{-i\mathbf{p} \cdot \mathbf{x}}. \end{aligned} \quad (177)$$

²¹ To verify the transformation, one can substitute the upper expression (177) into the lower one, take into account the spinor normalization $u^\dagger(\mathbf{p}, s) u(\mathbf{p}, s) = 2E_p$, that $\int \frac{d\mathbf{x} e^{i(\mathbf{p}' - \mathbf{p}) \cdot \mathbf{x}}}{(2\pi)^3} = \delta^3(\mathbf{p}' - \mathbf{p})$, and obtain the identity

$$\begin{aligned} \tilde{\psi}(\mathbf{p}, s) &= \int d\mathbf{x} \frac{u^\dagger(\mathbf{p}, s) e^{-i\mathbf{p} \cdot \mathbf{x}}}{\sqrt{2E_p}} \int \frac{d\mathbf{p}'}{(2\pi)^3} u(\mathbf{p}', s) \\ &\times \frac{e^{i\mathbf{p}' \cdot \mathbf{x}}}{\sqrt{2E_{p'}}} \tilde{\psi}(\mathbf{p}', s) = \int d\mathbf{p}' \int \frac{d\mathbf{x} e^{i(\mathbf{p}' - \mathbf{p}) \cdot \mathbf{x}}}{(2\pi)^3} \\ &\times \frac{u^\dagger(\mathbf{p}, s) u(\mathbf{p}', s)}{\sqrt{2E_p} \sqrt{2E_{p'}}} \tilde{\psi}(\mathbf{p}', s) \\ &= \frac{u^\dagger(\mathbf{p}, s) u(\mathbf{p}, s)}{2E_p} \tilde{\psi}(\mathbf{p}, s) = \tilde{\psi}(\mathbf{p}, s). \end{aligned}$$

These formulas can be expressed in more detail as follows:

$$\begin{aligned}\psi(\mathbf{x}) &\equiv \langle \mathbf{x} | \psi \rangle = \int \frac{d\mathbf{p}}{(2\pi)^3} u(\mathbf{p}, s) \frac{e^{i\mathbf{p}\mathbf{x}}}{\sqrt{2E_{\mathbf{p}}}} \frac{\langle \mathbf{p}, s | \psi \rangle}{\sqrt{2E_{\mathbf{p}}}} \\ &= \int \frac{d\mathbf{p}}{(2\pi)^3} u(\mathbf{p}, s) \frac{e^{i\mathbf{p}\mathbf{x}}}{\sqrt{2E_{\mathbf{p}}}} \tilde{\psi}(\mathbf{p}, s), \\ \tilde{\psi}(\mathbf{p}, s) &\equiv \frac{\langle \mathbf{p}, s | \psi \rangle}{\sqrt{2E_{\mathbf{p}}}} = \int d\mathbf{x} \frac{u^\dagger(\mathbf{p}, s) e^{-i\mathbf{p}\mathbf{x}}}{\sqrt{2E_{\mathbf{p}}}} \langle \mathbf{x} | \int d\mathbf{y} | \mathbf{y} \rangle \psi(\mathbf{y}) \\ &= \int d\mathbf{x} d\mathbf{y} \frac{u^\dagger(\mathbf{p}, s) e^{-i\mathbf{p}\mathbf{x}}}{\sqrt{2E_{\mathbf{p}}}} \langle \mathbf{x} | \mathbf{y} \rangle \psi(\mathbf{y}) = \int d\mathbf{x} \frac{u^\dagger(\mathbf{p}, s) e^{-i\mathbf{p}\mathbf{x}}}{\sqrt{2E_{\mathbf{p}}}} \psi(\mathbf{x}).\end{aligned}$$

10.1.2 Action of the translation operator. Of importance for this consideration is the expression for action of the single-particle state shift operator in the $\mathbf{p}$ -representation

$$e^{iq\hat{\mathbf{X}}} |\mathbf{p}\rangle = \frac{\sqrt{2E_{\mathbf{p}}}}{\sqrt{2E_{\mathbf{p}+\mathbf{q}}}} |\mathbf{p} + \mathbf{q}\rangle. \quad (178)$$

To clarify its meaning, we note that the spatial shift (translation) operator defined by the condition $\hat{\mathbf{T}}_a |\mathbf{x}\rangle = |\mathbf{x} - \mathbf{a}\rangle$ is given by the expression $\hat{\mathbf{T}}_a = e^{-i\mathbf{a}\hat{\mathbf{P}}}$, where $\hat{\mathbf{P}} = -i \frac{\partial}{\partial \mathbf{x}}$ [partial derivative] is the momentum operator. Similarly, the momentum space shift operator defined by the condition $\hat{\mathbf{T}}_q |\mathbf{p}\rangle = |\mathbf{p} + \mathbf{q}\rangle$ is given by the expression $\hat{\mathbf{T}}_q = e^{+iq\hat{\mathbf{X}}}$. Here the operator $\hat{\mathbf{X}} = -i \frac{\partial}{\partial \mathbf{p}}$ is the (three-dimensional) coordinate operator. Its action on the state $|\mathbf{x}\rangle$ with fixed coordinate is reduced to multiplication of this state by this coordinate, $\hat{\mathbf{X}} |\mathbf{x}\rangle = \mathbf{x} |\mathbf{x}\rangle$. Therefore, the operator $e^{iq\hat{\mathbf{X}}}$ transfers the state with momentum $\mathbf{p}$ to the state with momentum $\mathbf{p} + \mathbf{q}$, i.e., we have $e^{iq\hat{\mathbf{X}}} |\mathbf{p}\rangle = C |\mathbf{p} + \mathbf{q}\rangle$, and it remains to show that the normalization constant $C = \frac{\sqrt{2E_{\mathbf{p}}}}{\sqrt{2E_{\mathbf{p}+\mathbf{q}}}}$.

This, for example, can be seen from the normalization conditions $\langle \mathbf{p} | \mathbf{p}' \rangle = (2\pi)^3 2E_{\mathbf{p}} \delta(\mathbf{p} - \mathbf{p}')$ and $\langle \mathbf{p} + \mathbf{q} | \mathbf{p} + \mathbf{q} \rangle = (2\pi)^3 2E_{\mathbf{p}+\mathbf{q}}$, i.e., $\frac{\langle \mathbf{p} + \mathbf{q} | \mathbf{p} + \mathbf{q} \rangle}{2E_{\mathbf{p}+\mathbf{q}}} = \frac{\langle \mathbf{p} | \mathbf{p} \rangle}{2E_{\mathbf{p}}}$, from which follows $\frac{\sqrt{2E_{\mathbf{p}}}}{\sqrt{2E_{\mathbf{p}+\mathbf{q}}}} |\mathbf{p} + \mathbf{q}\rangle = e^{iq\hat{\mathbf{X}}} |\mathbf{p}\rangle$.

Another substantiation of expression (178) can be based on the observation that if $\mathbf{q}$ does not depend on $\mathbf{p}$, then $d\mathbf{p} = d(\mathbf{p} + \mathbf{q})$ and, since the action of the coordinate operator on the function $e^{i\mathbf{p}\mathbf{x}}$ has the form $\hat{\mathbf{X}} \cdot e^{i\mathbf{p}\mathbf{x}} = -i \frac{d}{d\mathbf{p}} e^{i\mathbf{p}\mathbf{x}} = \mathbf{x} e^{i\mathbf{p}\mathbf{x}}$, the action of the shift operator $e^{iq\hat{\mathbf{X}}}$ on this function yields $e^{iq\hat{\mathbf{X}}} \cdot e^{i\mathbf{p}\mathbf{x}} = e^{i\mathbf{x}(\mathbf{p}+\mathbf{q})}$.

Then, changing the integration variable in the formula for the nonrelativistic state (174), we have

$$\begin{aligned}\int \frac{d\mathbf{p} e^{i\mathbf{p}\mathbf{x}}}{(2\pi)^3} \frac{\langle \mathbf{p} |}{\sqrt{2E_{\mathbf{p}}}} &= \int \frac{d(\mathbf{p} + \mathbf{q}) e^{i\mathbf{x}(\mathbf{p}+\mathbf{q})}}{(2\pi)^3} \\ &\times \frac{\langle \mathbf{p} + \mathbf{q} |}{\sqrt{2E_{\mathbf{p}+\mathbf{q}}}} \int \frac{d\mathbf{p} e^{i\mathbf{p}\mathbf{x}}}{(2\pi)^3} \frac{\langle \mathbf{p} + \mathbf{q} | e^{iq\hat{\mathbf{X}}}}{\sqrt{2E_{\mathbf{p}+\mathbf{q}}}}.\end{aligned}$$

Coincidence of the integrand functions substantiates expression (178)²²

10.1.3. States containing n particles. Unit operators defined by Eqs. (171) are generalized to the case of the n -particle state in the form

$$\begin{aligned}\hat{I}_{\mathbf{p},n} &= \int \left(\prod_{i=1}^n \frac{d\mathbf{p}_i}{(2\pi)^3 2E_{\mathbf{p}_i}} \right) \frac{|\{p\}\rangle \langle \{p\}|}{n!}, \\ \hat{I}_{\mathbf{x},n} &= \int \left(\prod_{i=1}^n d\mathbf{x}_i \right) \frac{|\{x\}\rangle \langle \{x\}|}{n!}.\end{aligned} \quad (179)$$

In what follows, the symbols $\{p\}$ and $\{x\}$ are used as compact notation of the expressions $\{p\} \equiv (p_1 \dots p_n)$ and $\{x\} \equiv (x_1 \dots x_n)$. The so-called multiparticle bra vector $\langle \{x\} |$ has the form

$$\langle \{x\} | = \langle 0 | \psi_{m_1}(\mathbf{x}_1) \dots \psi_{m_n}(\mathbf{x}_n), \quad (180)$$

where $x_i = (\mathbf{x}_i, m_i)$, and m_i number spinor lines of fields $\psi(\mathbf{x}_i)$. By analogy with (175), wave functions in the momentum and coordinate representations for the n -particle state $|\psi\rangle$ can be obtained using unit operators (179)

$$\begin{aligned}|\psi\rangle &\equiv \hat{I}_{\mathbf{p},n} |\psi\rangle = \int \left(\prod_{i=1}^n d\tilde{\mathbf{p}}_i \right) \frac{\tilde{\psi}(\{p\})}{\sqrt{n!}} |\{p\}\rangle, \\ |\psi\rangle &\equiv \hat{I}_{\mathbf{x},n} |\psi\rangle = \int \left(\prod_{i=1}^n d\mathbf{x}_i \right) \frac{\psi(\{x\})}{\sqrt{n!}} |\{x\}\rangle,\end{aligned} \quad (181)$$

where

$$\tilde{\psi}(\{p\}) = \frac{\langle \{p\} | \psi \rangle}{\sqrt{n!} \prod_i \sqrt{2E_{\mathbf{p}_i}}} = \frac{\langle \{p\} | \psi \rangle}{\sqrt{n!}} \left(\prod_i \frac{1}{\sqrt{2E_{\mathbf{p}_i}}} \right), \quad (182)$$

$$\psi(\{x\}) = \frac{1}{\sqrt{n!}} \langle \{x\} | \psi \rangle,$$

and the designation

$$d\tilde{\mathbf{p}}_i \equiv \frac{d\mathbf{p}_i}{(2\pi)^3 \sqrt{2E_{\mathbf{p}_i}}}$$

²²Or, considering (174), we calculate the matrix element

$$\langle \mathbf{x} | e^{iq\hat{\mathbf{X}}} | \mathbf{p} \rangle = \int \frac{d\mathbf{p}'}{(2\pi)^3} e^{iq\hat{\mathbf{X}}} \frac{e'}{\sqrt{2E_{\mathbf{p}'}}} \langle \mathbf{p}' | \mathbf{p} \rangle. \text{ Since } e^{iq\hat{\mathbf{X}}} \cdot e^{i\mathbf{p}\mathbf{x}} = e^{i\mathbf{x}(\mathbf{p}+\mathbf{q})},$$

$$\text{then } \langle \mathbf{x} | e^{iq\hat{\mathbf{X}}} | \mathbf{p} \rangle = \int \frac{d\mathbf{p}'}{(2\pi)^3} \frac{e^{iq\mathbf{x}} e^{i\mathbf{p}'\mathbf{x}}}{\sqrt{2E_{\mathbf{p}'}}} (2\pi)^3 2E_{\mathbf{p}} \delta(\mathbf{p} - \mathbf{p}') =$$

$\sqrt{2E_{\mathbf{p}}} e^{i\mathbf{x}(\mathbf{p}+\mathbf{q})}$. Definition (172) of the scalar product can be written in the form $e^{i\mathbf{x}(\mathbf{p}+\mathbf{q})} = \frac{\langle \mathbf{x} | \mathbf{p} + \mathbf{q} \rangle}{\sqrt{2E_{\mathbf{p}+\mathbf{q}}}}$, from which follows (178).

is introduced for convenience. Multiparticle wave functions $\tilde{\psi}(\{p\})$ and $\psi(\{x\})$ are antisymmetric under permutation of all n particles, and they are also Fourier transforms of each other²³

$$\begin{aligned}\psi(\{x\}) &= \int \left(\prod_{i=1}^n d\tilde{\mathbf{p}}_i u_{m_i}(\mathbf{p}_i, s_i) e^{i\mathbf{p}_i \cdot \mathbf{x}_i} \right) \tilde{\psi}(\{p\}), \\ \tilde{\psi}(\{p\}) &= \int \left(\prod_{i=1}^n d\mathbf{x}_i \frac{u_{m_i}^\dagger(\mathbf{p}_i, s_i)}{\sqrt{2E_{\mathbf{p}_i}}} e^{-i\mathbf{p}_i \cdot \mathbf{x}_i} \right) \psi(\{x\}).\end{aligned}\quad (183)$$

Expressions (183) were derived using the following normalization conditions:

$$\begin{aligned}\langle \{x\} | \{p\} \rangle &= n! \prod_j u_{m_j}(\mathbf{p}_j, s_j) e^{i\mathbf{p}_j \cdot \mathbf{x}_j}, \\ \langle \{p\} | \{x\} \rangle &= \langle \{x\} | \{p\} \rangle^\dagger = n! \prod_j u_{m_j}^\dagger(\mathbf{p}_j, s_j) e^{-i\mathbf{p}_j \cdot \mathbf{x}_j}.\end{aligned}\quad (184)$$

Actually, from definitions (182) it follows

$$\begin{aligned}\psi(\{x\}) &= \frac{\langle \{x\} | \hat{I}_{\mathbf{p},n} | \Psi \rangle}{\sqrt{n!}} \\ &= \frac{1}{n!} \int \prod_i d\tilde{\mathbf{p}}_i \tilde{\psi}_n(\{p\}) \langle \{x\} | \{p\} \rangle \\ &= \int \prod_i d\tilde{\mathbf{p}}_i u_{m_i}(\mathbf{p}_i, s_i) e^{i\mathbf{p}_i \cdot \mathbf{x}_i} \tilde{\psi}(\{p\}), \\ \tilde{\psi}(\{p\}) &= \frac{\langle \{p\} | \hat{I}_{\mathbf{x},n} | \Psi \rangle}{\sqrt{n!}} \prod_i \frac{1}{\sqrt{2E_{\mathbf{p}_i}}} \\ &= \int \prod_i \frac{d\mathbf{x}_i}{\sqrt{2E_{\mathbf{p}_i}}} \frac{\psi(\{x\})}{n!} \langle \{p\} | \{x\} \rangle \\ &= \int \prod_i d\mathbf{x}_i \frac{u_{m_i}^\dagger(\mathbf{p}_i, s_i)}{\sqrt{2E_{\mathbf{p}_i}}} e^{-i\mathbf{p}_i \cdot \mathbf{x}_i} \psi(\{x\}).\end{aligned}\quad (185)$$

²³This is verified by direct substitution of $\psi(\{x\})$ into $\tilde{\psi}(\{p\})$ and the use of the spinor normalization condition

$$\begin{aligned}\tilde{\psi}(\{p\}) &\equiv (p_i, s_i, m_i) \\ &= \int \left(\prod_i d\mathbf{x}_i \frac{u_{m_i}^\dagger(\mathbf{p}_i, s_i)}{\sqrt{2E_{\mathbf{p}_i}}} e^{-i\mathbf{p}_i \cdot \mathbf{x}_i} \right) \\ &\times \int \left(\prod_j \frac{d\mathbf{p}'_j}{(2\pi)^3 \sqrt{2E_{\mathbf{p}'_j}}} u_{m_j}(\mathbf{p}'_j, s_j) e^{i\mathbf{p}'_j \cdot \mathbf{x}_j} \right) \tilde{\psi}(\{p'\}) \\ &= \int \prod_{i,j} \frac{d\mathbf{x}_i}{(2\pi)^3} \underbrace{e^{i\mathbf{x}_i \cdot (\mathbf{p}'_j - \mathbf{p}_i)}}_{\delta^3(\mathbf{p}_i - \mathbf{p}'_j)} \frac{d\mathbf{p}_j}{\sqrt{2E_{\mathbf{p}_i} 2E_{\mathbf{p}'_j}}} \\ &\times \underbrace{u_{m_i}^\dagger(\mathbf{p}_i, s_i) u_{m_j}(\mathbf{p}'_j, s_j)}_{2E_{\mathbf{p}'_j} \delta_{s_i, s_j} \delta_{m_i, m_j} \delta^3(\mathbf{p}_i - \mathbf{p}'_j)} \tilde{\psi}(\{p'\}) \\ &= (p'_j, s'_j, m'_j) = \tilde{\psi}(p_i, s_i, m_i).\end{aligned}$$

From the normalization condition for multiparticle functions $\langle \psi | \psi \rangle = 1$ normalization conditions for the wave functions $\tilde{\psi}(\{p\})$ and $\psi(\{x\})$ follow in the form

$$\begin{aligned}&\int \left(\prod_{i=1}^n \frac{d\mathbf{p}_i}{(2\pi)^3} \right) |\tilde{\psi}(\{p\})|^2 \\ &= \int \left(\prod_{i=1}^n d\mathbf{x}_i \right) |\psi(\{x\})|^2 = 1.\end{aligned}\quad (186)$$

This can be shown using the definition of the wave functions $|\psi\rangle$ according to (181). For their scalar product in the $\mathbf{p}$ -representation one has

$$\begin{aligned}\langle \psi | \psi \rangle &= \int \left(\prod_i d\tilde{\mathbf{p}}_i \right) \frac{\tilde{\psi}(\{p\})}{\sqrt{n!}} \langle \psi | \{p\} \rangle \\ &\equiv \int \left(\prod_i \frac{d\mathbf{p}_i}{(2\pi)^3 \sqrt{2E_{\mathbf{p}_i}}} \right) \frac{\tilde{\psi}(\{p\})}{\sqrt{n!}} \langle \psi | \{p\} \rangle.\end{aligned}$$

Considering the expression Hermitian conjugate to (182),

$$[\tilde{\psi}(\{p\})]^\dagger \sqrt{n!} \left(\prod_i \sqrt{2E_{\mathbf{p}_i}} \right) = \langle \{p\} | \Psi \rangle^\dagger = \langle \psi | \{p\} \rangle,$$

for $\langle \psi | \psi \rangle$ from the previous formula there follows

$$\begin{aligned}&\int \left(\prod_i \frac{d\mathbf{p}_i}{(2\pi)^3 \sqrt{2E_{\mathbf{p}_i}}} \right) \frac{\tilde{\psi}(\{p\})}{\sqrt{n!}} [\tilde{\psi}(\{p\})]^\dagger \sqrt{n!} \left(\prod_i \sqrt{2E_{\mathbf{p}_i}} \right) \\ &= \int \left(\prod_i \frac{d\mathbf{p}_i}{(2\pi)^3} \right) |\tilde{\psi}(\{p\})|^2 = 1.\end{aligned}$$

Using the definition of functions $|\psi\rangle$ (181), for the scalar product in the $\mathbf{x}$ -representation by considering the Hermitian conjugation to (182) in the form of $\langle \psi | \{x\} \rangle = \langle \{x\} | \Psi \rangle^\dagger = \psi(\{x\})^\dagger \sqrt{n!}$ one obtains

$$\begin{aligned}\langle \psi | \psi \rangle &= \int \left(\prod_{i=1}^n d\mathbf{x}_i \right) \frac{\psi(\{x\})}{\sqrt{n!}} \langle \psi | \{x\} \rangle \\ &= \int \left(\prod_i d\mathbf{x}_i \right) |\psi(\{x\})|^2 = 1.\end{aligned}\quad (187)$$

To make the picture complete, we will show that multiparticle states are normalized as follows:

$$\langle \{p\} | \{p'\} \rangle = n! \prod_i (2\pi)^3 2E_{\mathbf{p}_i} \delta^3(\mathbf{p}_i - \mathbf{p}'_i) \delta_{r_i, r'_i}.\quad (188)$$

Indeed, the state

$$\begin{aligned}|\{p'\}\rangle &\equiv \hat{I}_{\mathbf{p},n} |\{p'\}\rangle \\ &= \int \left(\prod_i \frac{d\mathbf{p}_i}{(2\pi)^3 2E_{\mathbf{p}_i}} \right) \frac{|\{p'\}\rangle}{n!} \langle \{p\} | \{p'\} \rangle\end{aligned}$$

should be equivalent (identically equal) to the state $|\{p'\}\rangle$. Therefore, the product $|\{p\}\rangle \langle \{p\} | \{p'\}\rangle$ also

should be proportional to $|\{p'\}\rangle$, but this is only possible if $\langle\{p|\{p'\}\rangle = C\delta(\{p\} - \{p'\})$. Therefore,

$$\begin{aligned}\hat{I}_{\mathbf{p},n}|\{p'\}\rangle &= \int \left(\prod_i^n \frac{d\mathbf{p}_i}{(2\pi)^3 2E_{\mathbf{p}_i}} \right) \\ &\times \frac{|\{p'\}\rangle}{n!} C\delta(\{p\} - \{p'\}) \\ &= |\{p'\}\rangle \left(\prod_i^n \frac{1}{(2\pi)^3 2E_{\mathbf{p}_i}} \right) \frac{C}{n!}\end{aligned}$$

coincides with $|\{p'\}\rangle$ at $C = n! \left(\prod_i^n (2\pi)^3 2E_{\mathbf{p}_i} \right)$. Hence follows (188).

10.1.4. Wave function of a nucleus as a bound composite system. On the basis of the formal relations from the previous section, the wave function $|P_n\rangle$, which completely describes the nucleus A in the n th quantum state with four-momentum P_n (see Fig. 2, left), can be written as a product of states A of free nucleons $|\{p\}\rangle$, which is weighted with the bound state wave function denoted for the present as $\tilde{\Psi}'_n(\{p\})$. In the (momentum) $\mathbf{p}$ -representation we have

$$|P_n\rangle = \int \left(\prod_{i=1}^A d\tilde{\mathbf{p}}_i \right) \frac{\tilde{\Psi}'_n(\{p\})}{\sqrt{A!}} |\{p\}\rangle. \quad (189)$$

The wave function $\tilde{\Psi}'_n(\{p\})$ is intended for describing the specifics of the internal structure of the nucleus and its motion as a whole with spin projection s and total three-momentum

$$\mathbf{p} = \sum_{i=1}^A \mathbf{p}_i. \quad (190)$$

Since the internal quantum state of the system of A nucleons cannot depend on the motion of their common center of mass, the wave function $\tilde{\Psi}'_n(\{p\})$ can be written as a product of the wave function $\tilde{\Psi}_n(\{p^\star\})$, which describes the internal structure of the nucleus in its center-of-mass system (its corresponding momenta are indicated by the superscript $\star$), and the wave function $\Phi_n(p)$, which describes motion of the nucleus as a whole with momentum $\mathbf{p}$ from (190) and nuclear spin projection s

$$\tilde{\Psi}_n(\{p\}) = \tilde{\Psi}_n(\{p^\star\})\Phi_n(p). \quad (191)$$

Here both global variables $\mathbf{p}$ and s are “hidden” in the argument $p = (\mathbf{p}, s)$ of the function $\Phi_n(p)$. Clearly, factorization (191) has the meaning only for nuclei with $A > 1$. In the laboratory coordinate system (if it does not coincide with the rest frame of the nucleus,

i.e., $\mathbf{p} \neq 0$) the three-momentum of the i th nucleon $\mathbf{p}_i$ is defined by the relation

$$\mathbf{p}_i = \mathbf{p}_i^\star + \mathbf{p}/A, \quad (192)$$

where $\mathbf{p}$, the total momentum of the nucleus, is given by formula (190), and the three-momentum of the i th nucleon in the center-of-mass system of the nucleus is denoted by $\mathbf{p}_i^\star$.

Considering (191), the Fock state (189) can be recast as

$$\begin{aligned}|P_n\rangle &= \int \left(\prod_i^A d\tilde{\mathbf{p}}_i \right) \frac{\tilde{\Psi}_n(\{p\})}{\sqrt{A!}} |\{p\}\rangle \\ &= \int \left(\prod_i^A d\tilde{\mathbf{p}}_i^\star \right) \frac{\tilde{\Psi}_n(\{p^\star\})}{\sqrt{A!}} \Phi_n(p) |\{p^\star\}\rangle.\end{aligned} \quad (193)$$

The wave function $\Phi_n(p)$ will be written in the form

$$\Phi_n(p) = (2\pi)^3 \sqrt{2P_n^0} \delta^3(\mathbf{p} - \mathbf{P}) \delta_{sS}, \quad (194)$$

which corresponds to the nucleus with total spin S defined by the three-momentum $\mathbf{P}$ and energy $P_n^0 = E_{\mathbf{p}} + \varepsilon_n$ in which the nuclear excitation energy ε_n is included.

We choose the normalization conditions for the nuclear states $|P_n\rangle$ from (193) in the same form as for the single-particle states $|\mathbf{p}, s\rangle$ in (165),

$$\langle P'_m | P_n \rangle = (2\pi)^3 2P_n^0 \delta^3(\mathbf{P} - \mathbf{P}') \delta_{nm}. \quad (195)$$

Then, the normalization condition for the nuclear states $|n\rangle$ describing the nucleus in its center-of-mass system will have the form

$$\begin{aligned}\langle m | n \rangle &\equiv \int \left(\prod_i^A \frac{d\mathbf{p}_i^\star}{(2\pi)^3} \right) \tilde{\Psi}_n(\{p^\star\}) \\ &\times \tilde{\Psi}_m^*(\{p^\star\}) (2\pi)^3 \delta^3 \left(\sum_{i=1}^A \mathbf{p}_i^\star \right) = \delta_{mn}.\end{aligned} \quad (196)$$

The delta function $\delta^3 \left(\sum_{i=1}^A \mathbf{p}_i^\star \right)$ decreases the number of independent momenta in (196) by unity. Note that at $\mathbf{P} = 0$ the wave functions $|n\rangle$ and $|P_n\rangle$ describe the same nuclear state but differ by the normalization condition. The explicit expression for $|n\rangle$ corresponding to normalization (196) can be purely formally written as

$$\begin{aligned}|n\rangle &= \int \left(\prod_{i=1}^A d\tilde{\mathbf{p}}_i^\star \right) \frac{\tilde{\Psi}_n(\{p^\star\})}{\sqrt{A!}} \\ &\times \left[(2\pi)^3 \delta^3 \left(\sum_{i=1}^A \mathbf{p}_i^\star \right) \right]^{1/2} |\{p^\star\}\rangle.\end{aligned} \quad (197)$$

With explicit consideration of $d\tilde{\mathbf{p}}_i^\star$, $|n\rangle =$

$$\int \left(\prod_i^A \frac{d\mathbf{p}_i^\star}{(2\pi)^3 \sqrt{2E_{\mathbf{p}_i^\star}}} \right) \frac{\tilde{\Psi}_n(\{p^\star\})}{\sqrt{A!}} \sqrt{(2\pi)^3 \delta^3 \left(\sum_{i=1}^A \mathbf{p}_i^\star \right)} |\{p^\star\}\rangle.$$

Let us make sure that normalization (195) is correct, i.e., we calculate the scalar product of nuclear states $\langle P'_m | P_n \rangle$. If $|P_n\rangle$ is given according to (193), one has, with redefinition of the integration variable,

$$\langle P'_m | = \int \left(\prod_j^A d\mathbf{p}_j^\star \right) \frac{\tilde{\Psi}_m^*(\{p^\star\})}{\sqrt{A!}} \Phi_m^*(p') \langle \{p^\star\} |. \quad (198)$$

Then,

$$\begin{aligned} \langle P'_m | P_n \rangle &= \int \left(\prod_j^A d\mathbf{p}_j^\star \right) \frac{\tilde{\Psi}_m^*(\{p^\star\})}{\sqrt{A!}} \\ &\quad \times \Phi_m^*(p') \langle \{p^\star\} | \int \left(\prod_j^A d\tilde{\mathbf{p}}_j^\star \right) \\ &\quad \times \frac{\tilde{\Psi}_n(\{p^\star\})}{\sqrt{A!}} \Phi_n(p) | \{p^\star\} \rangle \\ &= \int \prod_{j,i}^A \frac{d\mathbf{p}_j^\star d\mathbf{p}_i^\star}{(2\pi)^6 \sqrt{2E_{p_j^\star} 2E_{p_i^\star}}} \tilde{\Psi}_m^*(\{p^\star\}) \\ &\quad \times \tilde{\Psi}_n(\{p^\star\}) \Phi_m^*(p') \Phi_n(p) \frac{\langle \{p^\star\} | \{p^\star\} \rangle}{A!}. \end{aligned}$$

Using normalization condition (188) for the multi-particle state

$$\frac{\langle \{p^\star\} | \{p^\star\} \rangle}{A!} = \prod_k^A (2\pi)^3 2E_{p_k^\star} \delta^3(\mathbf{p}_k^\star - \mathbf{p}_k^\star) \delta_{r_k, r_k'}$$

and definition (194) for the functions $\Phi_m^*(p')$ and $\Phi_n(p)$, we go over to the expression

$$\begin{aligned} \langle P'_m | P_n \rangle &= \int \left(\prod_{j,i}^A \frac{d\mathbf{p}_j^\star d\mathbf{p}_i^\star}{(2\pi)^6 \sqrt{2E_{p_j^\star} 2E_{p_i^\star}}} \right) \\ &\quad \times \tilde{\Psi}_m^*(\{p^\star\}) \tilde{\Psi}_n(\{p^\star\}) [(2\pi)^6 \sqrt{2P_m^0 2P_n^0} \delta^3 \\ &\quad \times (\mathbf{p}' - \mathbf{P}_m') \delta^3(\mathbf{p} - \mathbf{P}_n)] \prod_k^A (2\pi)^3 2E_{p_k^\star} \delta^3(\mathbf{p}_k^\star - \mathbf{p}_k^\star). \end{aligned}$$

Removing integration over $\mathbf{p}_k^\star$ by using delta functions, we obtain

$$\begin{aligned} (2\pi)^6 \int \left(\prod_k^A (2\pi)^3 2E_{p_k^\star} \right) &\left(\prod_i^A \frac{d\mathbf{p}_i^\star}{(2\pi)^6 \sqrt{(2E_{p_i^\star})^2}} \right) \\ &\times \tilde{\Psi}_m^*(\{p^\star\}) \tilde{\Psi}_n(\{p^\star\}) \\ &\times \left[2P_n^0 \delta^3(\mathbf{P}_n - \mathbf{P}_m') \delta^3 \left(\sum_{i=1}^A \mathbf{p}_i^\star \right) \right], \end{aligned}$$

which is ultimately rearranged in the form

$$\begin{aligned} \langle P'_m | P_n \rangle &= \left[(2\pi)^3 2P_n^0 \delta^3(\mathbf{P}_n - \mathbf{P}_m') \right] \\ &\times \underbrace{\int \left(\prod_i^A \frac{d\mathbf{p}_i^\star}{(2\pi)^3} \right) \tilde{\Psi}_m^*(\{p^\star\}) \tilde{\Psi}_n(\{p^\star\}) (2\pi)^3 \delta^3 \left(\sum_{i=1}^A \mathbf{p}_i^\star \right)}_{\langle mn \rangle \equiv \delta_{mn}}, \end{aligned}$$

coinciding with (195). The equality

$$\begin{aligned} \delta^3(\mathbf{p}' - \mathbf{P}_m') \delta^3(\mathbf{p} - \mathbf{P}_n) \prod_k^A \delta^3(\mathbf{p}_k^\star - \mathbf{p}_k^\star) \\ = \delta^3(\mathbf{P}_n - \mathbf{P}_m') \delta^3 \left(\sum_{i=1}^A \mathbf{p}_i^\star \right), \end{aligned}$$

was used, which follows from $\prod_k^A \delta^3(\mathbf{p}_k^\star - \mathbf{p}_k^\star) = \prod_k^A \delta^3(\mathbf{p}_k - \mathbf{p}_k')$ because $\mathbf{p} = \sum_{j=1}^A \mathbf{p}_j$ and $\mathbf{p}' = \sum_{j=1}^A \mathbf{p}_j'$.

10.2. Notes to Calculation of the Scattering Amplitude

10.2.1. Formula for the hadronic current h_{mn}^μ . Let us derive expression (73) of the hadronic current h_{mn}^μ beginning with formula (70), $H_{mn}^\mu(P_n, P_m') = \langle P'_m | H^\mu(0) | P_n \rangle$. Wave functions of the nucleus in the final and initial states are given by formulas (193) and (198)

$$\begin{aligned} \langle P'_m | &= \int \left(\prod_j^A d\tilde{\mathbf{p}}_j^\star \right) \frac{\tilde{\Psi}_m^*(\{p^\star\})}{\sqrt{A!}} \Phi_m^*(p') \langle \{p^\star\} |, \\ | P_n \rangle &= \int \left(\prod_i^A d\tilde{\mathbf{p}}_i^\star \right) \frac{\tilde{\Psi}_n(\{p^\star\})}{\sqrt{A!}} \Phi_n(p) | \{p^\star\} \rangle. \end{aligned} \quad (199)$$

According to definition (194) (or (63)), the product of the “external” wave functions is

$$\begin{aligned} \Phi_m^*(p') \Phi_n(p) &= (2\pi)^3 \sqrt{2P_m'^0} \delta^3(\mathbf{p}' - \mathbf{P}_m') \\ &\times (2\pi)^3 \sqrt{2P_n^0} \delta^3(\mathbf{p} - \mathbf{P}_n), \end{aligned}$$

where $\mathbf{P}_n$ and $\mathbf{P}_m'$ are the three-momenta of the motion of the nucleus as a whole, and $\mathbf{p} = \sum_i^A \mathbf{p}_i$ and $\mathbf{p}' = \sum_i^A \mathbf{p}_i'$ are the variable arguments of the wave functions. From the definition of $H^\mu(x)$ (68) we have at $x = 0$

$$\begin{aligned} H_{mn}^\mu(P_n, P_m') &= \langle P'_m | H^\mu(0) | P_n \rangle \\ &\equiv \langle P'_m | \sum_k^A : \bar{\Psi}_k(0) O_k^\mu \Psi_k(0) : | P_n \rangle \\ &\quad \times [\delta^4(q + P_n - P_m')]. \end{aligned}$$

The presence of this δ^4 function is implicitly implied. From the definition of the single-particle fer-

mion quantum-field operators (167) we have for nucleons with the zero argument

$$\begin{aligned}\bar{\Psi}(0) &= \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{y}'}{\sqrt{2E_{\mathbf{y}'}}} \sum_{s=1,2} a_s^+(\mathbf{y}') \bar{u}(\mathbf{y}', s), \\ \hat{\Psi}(0) &= \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{y}}{\sqrt{2E_{\mathbf{y}}}} \sum_{r=1,2} a_r(\mathbf{y}) u(\mathbf{y}, r).\end{aligned}$$

Substituting these two formulas into the hadronic current $H_{mn}^\mu(P_n, P_m)$, we obtain

$$\begin{aligned}\langle P_m' | \frac{1}{(2\pi)^3} \sum_k \int \frac{d\mathbf{y}_k'}{\sqrt{2E_{\mathbf{y}_k'}}} \sum_{s=1,2} a_s^+(\mathbf{y}_k') \bar{u}(\mathbf{y}_k', s_k) \\ \times O_k^\mu \int \frac{d\mathbf{y}_k}{\sqrt{2E_{\mathbf{y}_k}}} \sum_{r=1,2} a_r(\mathbf{y}_k) u(\mathbf{y}_k, r_k) : | P_n \rangle \\ = \sum_k \sum_{s_k, r_k} \int \frac{d\mathbf{y}_k' d\mathbf{y}_k}{(2\pi)^3 \sqrt{2E_{\mathbf{y}_k'} 2E_{\mathbf{y}_k}}} \langle P_m' | a_s^+(\mathbf{y}_k') \\ \times [\bar{u}(\mathbf{y}_k', s_k) O_k^\mu u(\mathbf{y}_k, r_k) a_r(\mathbf{y}_k)] | P_n \rangle.\end{aligned}$$

Hence, considering the explicit form of $\langle P_m' |$ and introducing the notation $H_{mn}^\mu \equiv H_{mn}^\mu(P_n, P_m')$, we have

$$\begin{aligned}H_{mn}^\mu &= \sum_k \sum_{s_k, r_k} \int \frac{d\mathbf{y}_k' d\mathbf{y}_k}{(2\pi)^3 \sqrt{2E_{\mathbf{y}_k'} 2E_{\mathbf{y}_k}}} \\ &\times \int \left(\prod_j^A \tilde{\mathbf{p}}_j^\star \right) \frac{\tilde{\Psi}_m^*(\{p^\star\})}{\sqrt{A!}} \Phi_m^*(p') \\ &\times \int \left(\prod_j^A \tilde{\mathbf{p}}_j^\star d \right) \frac{\tilde{\Psi}_n(\{p^\star\})}{\sqrt{A!}} \Phi_n(p) \langle \{p^\star\} | \\ &\times a_s^+(\mathbf{y}_k') [\bar{u}(\mathbf{y}_k', s_k) O_k^\mu u(\mathbf{y}_k, r_k)] a_r(\mathbf{y}_k) | \{p^\star\} \rangle.\end{aligned} \quad (200)$$

The single-particle matrix element w_1^k from (202) is rearranged as

$$\begin{aligned}w_1^k &= \langle p_k^\star, s_{p'} | a_s^+(\mathbf{y}_k') [\bar{u}(\mathbf{y}_k', s_k) O_k^\mu u(\mathbf{y}_k, r_k)] a_r(\mathbf{y}_k) | \mathbf{p}_k^\star, r_p \rangle \\ &= \underbrace{(2\pi)^3 \sqrt{2E_{\mathbf{p}_k^\star} 2E_{\mathbf{p}_k^\star}}}_{\text{normalization}} \underbrace{\langle 0 | a_{s_{p'}}(\mathbf{p}_k^\star) | a_s^+(\mathbf{y}_k') [\bar{u}(\mathbf{y}_k', s_k) O_k^\mu u(\mathbf{y}_k, r_k)] a_r(\mathbf{y}_k) | \dots, a_{r_p}^+(\mathbf{p}_k^\star) | 0 \rangle}_{\delta^3(\mathbf{p}_k^\star - \mathbf{y}_k') \delta_{s, s_{p'}}} \\ &= (2\pi)^3 \sqrt{2E_{\mathbf{p}_k^\star} 2E_{\mathbf{p}_k^\star}} \delta^3(\mathbf{p}_k^\star - \mathbf{y}_k') \delta_{s, s_{p'}} \delta^3(\mathbf{p}_k^\star - \mathbf{y}_k) \delta_{r, r_p} \underbrace{[\bar{u}(\mathbf{y}_k', s_k) O_k^\mu u(\mathbf{y}_k, r_k)]}_{\times \delta^3(\mathbf{y}_k + \mathbf{q} - \mathbf{y}_k')} \underbrace{\langle 0 | 0 \rangle}_1.\end{aligned}$$

The single-particle normalization factor of the k th particle should have the multiplier $(2\pi)^{2 \times 3/2}$ for normalization condition (188) to be also valid for A particles (e.g., when the operator $O_k^\mu \equiv 1$). The delta function $\delta^3(\mathbf{y}_k + \mathbf{q} - \mathbf{y}_k')$ means that it is the k th nucleon

Here it is necessary to calculate the matrix element of the single particle operator of weak current over the multiparticle state of A free nucleons

$$w_A^k \equiv \langle \{p^\star\} | a_s^+(\mathbf{y}_k') [\bar{u}(\mathbf{y}_k', s_k) \times O_k^\mu u(\mathbf{y}_k, r_k)] a_r(\mathbf{y}_k) | \{p^\star\} \rangle. \quad (201)$$

Considering the definition of $\{p^\star\}$, the states $\langle \{p^\star\} |$ and $|\{p^\star\}\rangle$ can be written, with the active k th nucleon separated out, in the form

$$\begin{aligned}_A \langle \{p^\star\} | &= \langle 0 | (\dots, p_k^\star, \dots); \dots, a_{s_{p'}}(\mathbf{p}_k^\star), \dots |, \\ |\{p^\star\}\rangle_A &= |\dots, a_{r_p}^+(\mathbf{p}_k^\star), \dots; (\dots, p_k^\star, \dots) | 0 \rangle.\end{aligned}$$

Then w_A^k is written as a product of the normalization scalar product for $(A-1)$ nucleons (see condition (188))

$$\begin{aligned}_{A-1} \langle \{p^\star\} | |\{p^\star\}\rangle_{A-1} \\ \equiv \langle 0 | \dots, p_{k-1}^\star, p_{k+1}^\star, \dots; \dots, p_{k-1}^\star, p_{k+1}^\star, \dots | 0 \rangle \\ = (A-1)! \left(\prod_{l \neq k}^{A-1} (2\pi)^3 2E_{\mathbf{p}_l^\star} \delta^3(\mathbf{p}_l^\star - \mathbf{p}_{l'}^\star) \delta_{r_l, r_{l'}} \right)\end{aligned}$$

and the matrix element of weak current over the single-nucleon k th state

$$w_1^k \equiv \langle \mathbf{p}_k^\star, s_{p'} | a_s^+(\mathbf{y}_k') \times [\bar{u}(\mathbf{y}_k', s_k) O_k^\mu u(\mathbf{y}_k, r_k)] a_r(\mathbf{y}_k) | \mathbf{p}_k^\star, r_p \rangle. \quad (202)$$

That is, expression (201) takes the form

$$\begin{aligned}w_A^k &= \langle 0 | (\dots, p_k^\star, \dots); \dots, a_{s_{p'}}(\mathbf{p}_k^\star), \\ &\dots | a_s^+(\mathbf{y}_k') [\bar{u}(\mathbf{y}_k', s_k) O_k^\mu u(\mathbf{y}_k, r_k)] a_r(\mathbf{y}_k) | \dots, \\ &a_{r_p}^+(\mathbf{p}_k^\star), \dots; (\dots, p_k^\star, \dots) | 0 \rangle\end{aligned} \quad (203)$$

with which the neutrino interacts, changing its momentum from $\mathbf{y}_k$ to $\mathbf{y}_k' = \mathbf{q} + \mathbf{y}_k$.

Considering this expression for w_1^k , the matrix element of weak current over the state of A free nucleons (332) takes the form

$$w_A^k = (A-1)! \left(\prod_{l \neq k}^{A-1} (2\pi)^3 2E_{\mathbf{p}_l^*} \delta^3(\mathbf{p}_l^* - \mathbf{p}_l'^*) \delta_{\eta_l, \eta_l'} \right) [\bar{u}(\mathbf{y}'_k, s_k) O_k^\mu u(\mathbf{y}_k, r_k)] \delta^3(\mathbf{y}_k + \mathbf{q} - \mathbf{y}'_k) \\ \times (2\pi)^3 \sqrt{2E_{\mathbf{p}_k^*} 2E_{\mathbf{p}_k'^*}} \delta^3(\mathbf{p}_k^* - \mathbf{y}'_k) \delta_{s_k, s_{p'}} \delta^3(\mathbf{p}_l^* - \mathbf{y}_k) \delta_{r_k, r_p}.$$

Substituting this expression to the formula for hadronic current (200) and considering that the active k th nucleon can be at any of A places in the nucleus, which gives the factor A , we obtain

$$H_{mn}^\mu = \sum_k^A \sum_{s_k, r_k} \int \frac{d\mathbf{y}'_k d\mathbf{y}_k}{(2\pi)^3 \sqrt{2E_{\mathbf{y}_k} 2E_{\mathbf{y}'_k}}} \int \left(\prod_{j, l}^A \frac{d\mathbf{p}_j^* d\mathbf{p}_l^*}{(2\pi)^6 \sqrt{2E_{\mathbf{p}_j^*} 2E_{\mathbf{p}_l^*}}} \right) \frac{\tilde{\Psi}_m^*(\{p^*\}) \tilde{\Psi}_n(\{p^*\})}{A!} \Phi_m^*(p') \Phi_n(p) A(A-1)! \\ \times \left(\prod_{l \neq k}^{A-1} (2\pi)^3 2E_{\mathbf{p}_l^*} \delta^3(\mathbf{p}_l^* - \mathbf{p}_l'^*) \delta_{\eta_l, \eta_l'} \right) [\bar{u}(\mathbf{y}'_k, s_k) O_k^\mu u(\mathbf{y}_k, r_k)] \delta^3(\mathbf{y}_k + \mathbf{q} - \mathbf{y}'_k) (2\pi)^3 \sqrt{2E_{\mathbf{p}_k^*} 2E_{\mathbf{p}_k'^*}} \delta^3(\mathbf{p}_k^* - \mathbf{y}'_k) \\ \times \delta_{s_{\mathbf{y}_k}, s_{\mathbf{p}_k}} \delta^3(\mathbf{p}_k^* - \mathbf{y}_k) \delta_{s_{\mathbf{y}_k}, s_{\mathbf{p}_k}} = \sum_k^A \int \left(\prod_{j, l}^A \frac{d\mathbf{p}_j^* d\mathbf{p}_l^*}{(2\pi)^6 \sqrt{4E_{\mathbf{p}_j^*} E_{\mathbf{p}_l^*}}} \right) \left[\prod_{l \neq k}^{A-1} (2\pi)^3 2E_{\mathbf{p}_l^*} \delta^3(\mathbf{p}_l^* - \mathbf{p}_l'^*) \delta_{\eta_l, \eta_l'} \right] \tilde{\Psi}_m^*(\{p^*\}) \tilde{\Psi}_n(\{p^*\}) \\ \times \Phi_m^*(p') \Phi_n(p) \int \frac{d\mathbf{y}'_k d\mathbf{y}_k \delta^3(\mathbf{p}_k^* - \mathbf{y}'_k) \delta^3(\mathbf{p}_k^* - \mathbf{y}_k)}{\sqrt{4E_{\mathbf{y}_k} E_{\mathbf{y}'_k}}} \sqrt{4E_{\mathbf{p}_k^*} E_{\mathbf{p}_k'^*}} [\bar{u}(\mathbf{y}'_k, s_k) O_k^\mu u(\mathbf{y}_k, r_k)] \delta^3(\mathbf{y}_k + \mathbf{q} - \mathbf{y}'_k).$$

Integration over $d\mathbf{y}'_k d\mathbf{y}_k$ in this equation (and using of $\sum_{s_k, r_k, s_{\mathbf{y}_k}, s_{\mathbf{p}_k}} \delta_{s_{\mathbf{y}_k}, s_{\mathbf{p}_k}} = 1$) yields

$$H_{mn}^\mu = \sum_k^A \int \left(\prod_{j, l}^A \frac{d\mathbf{p}_j^* d\mathbf{p}_l^*}{(2\pi)^6 \sqrt{4E_{\mathbf{p}_j^*} E_{\mathbf{p}_l^*}}} \right) \left[\prod_{l \neq k}^{A-1} (2\pi)^3 2E_{\mathbf{p}_l^*} \delta^3(\mathbf{p}_l^* - \mathbf{p}_l'^*) \delta_{\eta_l, \eta_l'} \right] \tilde{\Psi}_m^*(\{p^*\}) \tilde{\Psi}_n(\{p^*\}) \\ \times \frac{\sqrt{4E_{\mathbf{p}_k^*} E_{\mathbf{p}_k'^*}}}{\sqrt{4E_{\mathbf{p}_k^*} E_{\mathbf{p}_k'^*}}} [\bar{u}(\mathbf{p}_k^*, s_k) O_k^\mu u(\mathbf{p}_k^*, r_k)] \delta^3(\mathbf{p}_k^* + \mathbf{q} - \mathbf{p}_k^*) [\Phi_m^*(p') \Phi_n(p)].$$

Now it is possible to integrate over $d\mathbf{p}_j^*$ (except for $\mathbf{p}_k^*$) after recasting H_{mn}^μ as

$$H_{mn}^\mu = \sum_k^A \int \left(\prod_{j, l \neq k}^A \prod_{l \neq k}^{A-1} \frac{(2\pi)^3 d\mathbf{p}_j^* d\mathbf{p}_l^* \delta^3(\mathbf{p}_l^* - \mathbf{p}_l'^*)}{(2\pi)^6 \sqrt{E_{\mathbf{p}_j^*} E_{\mathbf{p}_l^*}}} E_{\mathbf{p}_l^*} \right) \frac{d\mathbf{p}_k^* d\mathbf{p}_k^* \left\{ \delta^3(\mathbf{p}_k^* + \mathbf{q} - \mathbf{p}_k^*) \right\}}{(2\pi)^6 \sqrt{4E_{\mathbf{p}_k^*} E_{\mathbf{p}_k'^*}}} \tilde{\Psi}_m^*(\{p^*\}) \tilde{\Psi}_n(\{p^*\}) \\ \times [\Phi_m^*(p') \Phi_n(p)] [\bar{u}(\mathbf{p}_k^*, s_{p_k'}) O_k^\mu u(\mathbf{p}_k^*, r_{p_k})] = \sum_k^A \int \left(\prod_{j, l \neq k}^A \frac{d\mathbf{p}_j^* d\mathbf{p}_l^* \delta^3(\mathbf{p}_l^* - \mathbf{p}_l'^*)}{(2\pi)^3 \sqrt{E_{\mathbf{p}_j^*} E_{\mathbf{p}_l^*}}} E_{\mathbf{p}_l^*} \right) \\ \times \tilde{\Psi}_m^*(\{p^*\}) \tilde{\Psi}_n(\{p^*\}) \left[\bar{u}(\mathbf{p}_k^*, s_{p_k'}) O_k^\mu u(\mathbf{p}_k^*, r_{p_k}) \right] \left[(2\pi)^6 \sqrt{4P_m^{0'} P_n^0} \delta^3(\mathbf{p}' - \mathbf{P}_m') \delta^3(\mathbf{p} - \mathbf{P}_n) \right] \frac{d\mathbf{p}_k^* d\mathbf{p}_k^* \left\{ \delta^3(\mathbf{p}_k^* + \mathbf{q} - \mathbf{p}_k^*) \right\}}{(2\pi)^6 \sqrt{4E_{\mathbf{p}_k^*} E_{\mathbf{p}_k'^*}}},$$

and introducing the designation

$$\tilde{\Psi}_m^*(\{p_\star^{(k)}\}) \equiv \tilde{\Psi}_m^*(\{p^*, p_k^* \neq \mathbf{p}_k^*, s_{p_k'} \neq r_{p_k}\}),$$

where the argument $\{p_\star^{(k)}\}$ coincides with $\{p^*\}$, except that at the k th place in $\{p_\star^{(k)}\}$ instead of the pair $(\mathbf{p}_k^*, r_{p_k})$ there is another pair $(\mathbf{p}_k^*, s_{p_k'}) \neq (\mathbf{p}_k^*, r_{p_k})$, we obtain

$$H_{mn}^\mu = \sum_k^A \int \left(\prod_{i \neq k}^A \frac{E_{\mathbf{p}_i^*} d\mathbf{p}_i^*}{(2\pi)^3 \sqrt{E_{\mathbf{p}_i^*} E_{\mathbf{p}_i^*}}} \right) \tilde{\Psi}_m^*(\{p_\star^{(k)}\}) \tilde{\Psi}_n(\{p^*\}) [\bar{u}(\mathbf{p}_k^*, s_{p_k'}) O_k^\mu u(\mathbf{p}_k^*, r_{p_k})] \\ \times [\sqrt{4P_m^{0'} P_n^0} \delta^3(\mathbf{p}' - \mathbf{P}_m') \delta^3(\mathbf{p} - \mathbf{P}_n)] \frac{d\mathbf{p}_k^* d\mathbf{p}_k^* \delta^3(\mathbf{p}_k^* + \mathbf{q} - \mathbf{p}_k^*)}{\sqrt{4E_{\mathbf{p}_k^*} E_{\mathbf{p}_k'^*}}}.$$

Replacement of $d\mathbf{p}_k^\star$ with $(2\pi)^3 \frac{d\mathbf{p}_k^\star}{(2\pi)^3}$ and integration with $\delta^3(\mathbf{p}_k^\star + \mathbf{q} - \mathbf{p}_k'^\star)$ over $d\mathbf{p}_k^\star$ results in the following expression for hadronic current (200):

$$\begin{aligned} H_{mn}^\mu(P_m', P_n) &= (2\pi)^3 \sum_k \int \left(\prod_i \frac{d\mathbf{p}_i^\star}{(2\pi)^3} \right) \\ &\times \frac{[\bar{u}(\mathbf{p}_k^\star = \mathbf{p}_k^\star + \mathbf{q}, s_{p_k}) O_k^\mu u(\mathbf{p}_k^\star, r_{p_k})]}{\sqrt{4E_{p_k^\star = \mathbf{p}_k^\star + \mathbf{q}} E_{p_k^\star}}} \\ &\times \tilde{\Psi}_m^*(\{p_\star^{(k)}\}, \mathbf{p}_k^\star = \mathbf{p}_k^\star + \mathbf{q}) \tilde{\Psi}_n(\{p_\star\}) \\ &\times \sqrt{4P_m^0 P_n^0} \delta^3(\mathbf{p}' - \mathbf{P}_m') \delta^3(\mathbf{p} - \mathbf{P}_n). \end{aligned} \quad (204)$$

It only remains to examine the product $\delta^3(\mathbf{p}' - \mathbf{P}_m') \delta^3(\mathbf{p} - \mathbf{P}_n)$, where $\mathbf{p} = \sum_i^A \mathbf{p}_i$ and $\mathbf{p}' = \sum_i^A \mathbf{p}_i'$, i.e., to show that the product of the “external” functions $\Phi_m^*(p') \Phi_n(p)$ can be written as

$$\begin{aligned} \Phi_m^*(p') \Phi_n(p) &= (2\pi)^6 2\sqrt{P_m^0 P_n^0} \\ &\times \delta^3(\mathbf{q} + \mathbf{P}_n - \mathbf{P}_m') \delta^3\left(\sum_{i=1}^A \mathbf{p}_i^\star\right). \end{aligned} \quad (205)$$

Indeed, if the neutrino interacted with the k th nucleon (in the center-of-mass system of the nucleus) and transferred the momentum $\mathbf{q}$ to it, then $\mathbf{p}_k^\star = \mathbf{p}_k^\star + \mathbf{q}$; on the other (“external”) side, that particular transferred momentum $\mathbf{q}$ is eventually taken by the entire nucleus, i.e., $\mathbf{q} = \mathbf{P}_n - \mathbf{P}_m'$. If before interaction the nucleus was at rest ($\mathbf{P}_n = 0$), $\delta^3(\mathbf{p} - \mathbf{P}_n)$ passes into $\delta^3(\mathbf{p})$, giving $\mathbf{p} = 0$, but in the rest frame nucleus $\mathbf{p}_i = \mathbf{p}_i^\star$ by definition, and then $\mathbf{p} \equiv \sum_i^A \mathbf{p}_i = \mathbf{p}^\star \equiv \sum_{i=1}^A \mathbf{p}_i^\star = 0$, which is equivalent to having $\delta^3(\mathbf{p}^\star) \equiv \delta^3(\sum_{i=1}^A \mathbf{p}_i^\star)$. Further it is clear that if $\mathbf{P}_n = 0$, then $\mathbf{q} = \mathbf{P}_m'$, to which $\delta^3(\mathbf{q} - \mathbf{P}_m')$ corresponds. Generalization of this delta function to the case of $\mathbf{P}_n > 0$ yields $\delta^3(\mathbf{q} + \mathbf{P}_n - \mathbf{P}_m')$. In other words, $\delta^3(\mathbf{p}' - \mathbf{P}_m') \delta^3(\mathbf{p} - \mathbf{P}_n) \delta^3 \times (\sum_{i=1}^A \mathbf{p}_i^\star) \delta^3(\mathbf{q} + \mathbf{P}_n - \mathbf{P}_m')$. Therefore, the ultimate expression for hadronic current (39) takes the form

$$\begin{aligned} H_{mn}^\mu(P_m', P_n) &= \langle P_m' | H^\mu(0) | P_n \rangle \\ &= 2\sqrt{P_m^0 P_n^0} \times h_{mn}^\mu(\mathbf{q}) \delta^3(\mathbf{q} + \mathbf{P}_n - \mathbf{P}_m'), \end{aligned} \quad (206)$$

where, according to (73), for the hadronic current in the nuclear rest frame we have

$$\begin{aligned} h_{mn}^\mu &= \sum_k \int \prod_i^A \frac{d\mathbf{p}_i^\star}{(2\pi)^3} \frac{\bar{u}(\mathbf{p}_k^\star + \mathbf{q}, s_{p_k}) O_k^\mu u(\mathbf{p}_k^\star, r_{p_k})}{\sqrt{4E_{p_k^\star} E_{p_k^\star + \mathbf{q}}}} \\ &\times \tilde{\Psi}_m^*(\{p_\star^{(k)}\}, \mathbf{p}_k^\star + \mathbf{q}) \tilde{\Psi}_n(\{p_\star\}) (2\pi)^3 \delta^3(\sum_{i=1}^A \mathbf{p}_i^\star). \end{aligned} \quad (207)$$

Arguments of spinor functions $\mathbf{p}_i^\star$ are taken in the rest frame of the nucleus. Note that if one puts $O_k^\mu \equiv 1$ and $\mathbf{q} = 0$ in this expression, h_{mn}^μ should coincide with Eq. (196) for normalization of $\langle m | n \rangle$, because

$$\begin{aligned} \bar{u}(\mathbf{p}_k + \mathbf{q}, s) u(\mathbf{p}_k, r) &= \bar{u}(\mathbf{p}_k, s) u(\mathbf{p}_k, r) \\ &= 2E_{p_k} \delta(\mathbf{p}_k + \mathbf{q} - \mathbf{p}_k) \delta_{sr}. \end{aligned}$$

10.2.2. Formula for the hadronic structure f_{mn}^k . Concerning derivation of formula (78) for the hadronic structure $f_{mn}^k = \langle m | e^{iq\hat{\mathbf{x}}_k} | n \rangle$, we note that the operator $e^{iq\hat{\mathbf{x}}}$ increases the three-momentum $\mathbf{p}$ of the state $|\mathbf{p}\rangle$ by $\mathbf{q}$, transferring the state $|\mathbf{p}\rangle$ to the state $|\mathbf{p} + \mathbf{q}\rangle$ through the operation

$$e^{iq\hat{\mathbf{x}}} |\mathbf{p}\rangle = \frac{\sqrt{2E_{\mathbf{p}}}}{\sqrt{2E_{\mathbf{p}+\mathbf{q}}}} |\mathbf{p} + \mathbf{q}\rangle, \quad (208)$$

which is justified in appendix 10.1.2. Below, calculation of the matrix element $\langle m | e^{iq\hat{\mathbf{x}}_k} | n \rangle$ using (208) and (197) is given.

First, we calculate the matrix element $\langle P_m' | e^{iq\hat{\mathbf{x}}_k} | P_n \rangle$. Considering the initial $|P_n\rangle$ and final $\langle P_m'|$ nuclear states given in the form (199), we have

$$\begin{aligned} \langle P_m' | e^{iq\hat{\mathbf{x}}_k} | P_n \rangle &= \int \left(\prod_j^A d\mathbf{p}_j^\star \right) \frac{\tilde{\Psi}_m^*(\{p_\star'\})}{\sqrt{A!}} \Phi_m^*(p') \\ &\times \langle \{p_\star'\} | e^{iq\hat{\mathbf{x}}_k} \int \left(\prod_i^A d\mathbf{p}_i^\star \right) \frac{\tilde{\Psi}_n(\{p_\star\})}{\sqrt{A!}} \Phi_n(p) | \{p_\star\} \rangle \\ &= \int \left(\prod_{j,i}^A \frac{d\mathbf{p}_j^\star d\mathbf{p}_i^\star}{(2\pi)^6 \sqrt{2E_{p_j^\star} 2E_{p_i^\star}}} \right) \frac{\tilde{\Psi}_m^*(\{p_\star'\}) \tilde{\Psi}_n(\{p_\star\})}{A!} \\ &\times \Phi_m^*(p') \Phi_n(p) \underbrace{\langle \{p_\star'\} | e^{iq\hat{\mathbf{x}}_k} | \{p_\star\} \rangle}_{Q_k}. \end{aligned} \quad (209)$$

The matrix element of the k th nucleon shift operator $e^{iq\hat{\mathbf{x}}_k}$ over the multiparticle state of free nucleons can be written as a product of the normalization of the

spectator nucleons and the matrix element of the action of the operator $e^{iq\hat{X}_k}$ on the k th nucleon

$$Q_k \equiv \langle \{p_\star'\} | e^{iq\hat{X}_k} | \{p_\star\} \rangle = \langle \dots, p_k^\star, \dots | e^{iq\hat{X}_k} | \dots, p_k^\star, \dots \rangle \\ = \langle \dots, p_{k-1}^\star, p_{k+1}^\star, \dots | \dots, p_{k-1}^\star, p_{k+1}^\star, \dots \rangle \\ \times \langle p_k^\star | e^{iq\hat{X}_k} | p_k^\star \rangle \delta_{r_k, r_k'} \quad (210)$$

Note that if in (210) we set $\mathbf{q} = 0$, then $e^{iq\hat{X}_k} = 1$, and the matrix element $\langle \{p_\star'\} | e^{iq\hat{X}_k} | \{p_\star\} \rangle$ turns into normalization condition (188) for the multiparticle state of a free nucleon in the rest frame of the nucleus,

$$\langle \{p_\star'\} | \{p_\star\} \rangle = A! \left(\prod_l^A (2\pi)^3 2E_{p_l^\star} \delta^3(\mathbf{p}_l^\star - \mathbf{p}_l^\star) \delta_{r_l, r_l'} \right).$$

Therefore, at $\mathbf{q} = 0$ the formula for Q_k should take the form that passes into this very formula, i.e., using (208) and normalization of single-particle states, Eq. (210) can be written as

$$Q_k = A(A-1)! \left(\prod_{l \neq k}^{A-1} (2\pi)^3 2E_{p_l^\star} \delta^3(\mathbf{p}_l^\star - \mathbf{p}_l^\star) \delta_{r_l, r_l'} \right) \\ \times \delta_{r_k, r_k'} 2\sqrt{E_{p_k^\star} E_{p_k^\star + \mathbf{q}}} (2\pi)^3 \delta^3(\mathbf{p}_k^\star + \mathbf{q} - \mathbf{p}_k^\star), \quad (211)$$

where it is taken into consideration that the k th nucleon can be in any of A possible positions in the nucleus A , which gives the additional permutation factor A .

Substituting (211) into nuclear matrix element (209) (and omitting all $\delta_{r_k, r_k'}, \delta_{r_l, r_l'}$, since the shift operator does not affect the spin), we obtain

$$\langle P_m' | e^{iq\hat{X}_k} | P_n \rangle = \int \left(\prod_{j,i}^A \frac{d\mathbf{p}_j^\star d\mathbf{p}_i^\star}{(2\pi)^6 \sqrt{2E_{p_j^\star} 2E_{p_i^\star}}} \right) \\ \times \left(\prod_{j \neq k}^A (2\pi)^3 2E_{p_j^\star} \delta^3(\mathbf{p}_j^\star - \mathbf{p}_j^\star) \right) \\ \times \sqrt{2E_{p_k^\star} 2E_{p_k^\star + \mathbf{q}}} (2\pi)^3 \delta^3(\mathbf{p}_k^\star + \mathbf{q} - \mathbf{p}_k^\star) \\ \times \tilde{\Psi}_m^*(\{p_\star'\}) \tilde{\Psi}_n(\{p_\star\}) [\Phi_m^*(p') \Phi_n(p)].$$

After integrating over $d\mathbf{p}_j^\star$ (without $d\mathbf{p}_k^\star$) and substituting Eq. (205) for $[\Phi_m^*(p') \Phi_n(p)]$, we have

$$\langle P_m' | e^{iq\hat{X}_k} | P_n \rangle = \int \left(\prod_{i \neq k}^A \frac{d\mathbf{p}_i^\star}{(2\pi)^3 2E_{p_i^\star}} \prod_{l \neq k}^A 2E_{p_l^\star} \right) \\ \times \frac{(2\pi)^3 d\mathbf{p}_k^\star \sqrt{E_{p_k^\star} E_{p_k^\star + \mathbf{q}}}}{(2\pi)^6 \sqrt{E_{p_k^\star} E_{p_k^\star}}} d\mathbf{p}_k^\star \delta^3(\mathbf{p}_k^\star + \mathbf{q} - \mathbf{p}_k^\star) \\ \times \tilde{\Psi}_m^*(\{p_\star'\}, \mathbf{p}_k^\star \neq \mathbf{p}_k^\star) \tilde{\Psi}_n(\{p_\star\}) \\ \times \left[(2\pi)^6 2\sqrt{P_m^{0'} P_n^0} \delta^3(\mathbf{q} + \mathbf{P}_n - \mathbf{P}_m') \delta^3 \left(\sum_{i=1}^A \mathbf{p}_i^\star \right) \right].$$

Due to the delta function $\delta^3(\mathbf{p}_k^\star + \mathbf{q} - \mathbf{p}_k^\star)$, integration over $d\mathbf{p}_k^\star$ replaces $\mathbf{p}_k^\star$ with $\mathbf{p}_k^\star + \mathbf{q}$ and converts the multiplier $\frac{d\mathbf{p}_k^\star \sqrt{E_{p_k^\star} E_{p_k^\star + \mathbf{q}}}}{(2\pi)^3 \sqrt{E_{p_k^\star} E_{p_k^\star}}}$ into $\frac{d\mathbf{p}_k^\star}{(2\pi)^3}$. As a result, considering the assumption of factorization of the spin and momentum dependences (75),

$\tilde{\Psi}_n(\{p_\star'\}) = \tilde{\Psi}_n(\{p_\star\}) \chi_n(\{r'\})$, and spin diagonality, $\chi_m^*(\{r'\}) \chi_n(\{r'\}) = \delta_{mn}$, we arrive at the expression for matrix element (209)

$$\langle P_m' | e^{iq\hat{X}_k} | P_n \rangle = (2\pi)^6 2\sqrt{P_m^{0'} P_n^0} \delta^3(\mathbf{q} + \mathbf{P}_n - \mathbf{P}_m') \\ \times \int \left(\prod_i^A \frac{d\mathbf{p}_i^\star}{(2\pi)^3} \right) \tilde{\Psi}_m^*(\{p_\star^{(k)}\}) \tilde{\Psi}_n(\{p_\star\}) \delta^3 \left(\sum_{i=1}^A \mathbf{p}_i^\star \right).$$

Considering that

$$\langle m | e^{iq\hat{X}_k} | n \rangle \equiv \int \left(\prod_i^A \frac{d\mathbf{p}_i^\star}{(2\pi)^3} \right) \tilde{\Psi}_m^*(\{p_\star^{(k)}\}) \\ \times \tilde{\Psi}_n(\{p_\star\}) (2\pi)^3 \delta^3 \left(\sum_{i=1}^A \mathbf{p}_i^\star \right), \quad (212)$$

for matrix element (209) there arises the expression

$$\langle P_m' | e^{iq\hat{X}_k} | P_n \rangle \\ = (2\pi)^3 \left[2\sqrt{P_m^{0'} P_n^0} \delta^3(\mathbf{P}_n + \mathbf{q} - \mathbf{P}_m') \right] \langle m | e^{iq\hat{X}_k} | n \rangle$$

relating the nuclear matrix element of the shift operator in the nuclear rest frame to its analog in another frame of reference, where the initial momentum of the nucleus is $\mathbf{P}_n$.

Let us calculate matrix element (78) $f_{mn}^k \equiv \langle m | e^{iq\hat{X}_k} | n \rangle$. By definition, it is given in the rest frame of the nucleus, where

$$\mathbf{p} \equiv \sum_j^A \mathbf{p}_j = \sum_j^A \mathbf{p}_j^\star = 0.$$

For the nuclear states $|n\rangle$ given in the nuclear rest frame we will (temporarily) consider, on the basis of Eq. (197),

$$|n\rangle = \int \left(\prod_{i=1}^A d\mathbf{p}_i^\star \right) \frac{\tilde{\Psi}_n(\{p_\star'\})}{\sqrt{A!}} \\ \times \left[(2\pi)^3 \delta^3 \left(\sum_{i=1}^A \mathbf{p}_i^\star \right) \right]^{1/2} |\{p_\star'\}\rangle,$$

that the final and initial quantum states of the nucleus are

$$\langle m | = \int \phi_m^*(0) \left(\prod_j^A d\mathbf{p}_j^\star \right) \frac{\tilde{\Psi}_m^*(\{p_\star'\})}{\sqrt{A!}} \langle \{p_\star'\} |, \\ |n\rangle = \int \phi_n(0) \left(\prod_i^A d\mathbf{p}_i^\star \right) \frac{\tilde{\Psi}_n(\{p_\star\})}{\sqrt{A!}} |\{p_\star\}\rangle. \quad (213)$$

Then matrix element (78) takes the form

$$\begin{aligned}
 f_{mn}^k &= \int \phi_m^*(0) \phi_n(0) \left(\prod_j^A d\tilde{\mathbf{p}}_j^* \right) \frac{\tilde{\Psi}_m^*(\{p_\star'\})}{\sqrt{A!}} \langle \{p_\star'\} | \\
 &\times \int \left(\prod_i^A d\tilde{\mathbf{p}}_i^* \right) e^{iq\hat{\mathbf{x}}_k} \frac{\tilde{\Psi}_n(\{p_\star\})}{\sqrt{A!}} | \{p_\star\} \rangle \\
 &= \int C \prod_{j,i}^A \frac{d\mathbf{p}_j^* d\mathbf{p}_i^*}{(2\pi)^6 \sqrt{2E_{p_j^*} 2E_{p_i^*}}} \\
 &\times \frac{\tilde{\Psi}_m^*(\{p_\star'\}) \tilde{\Psi}_n(\{p_\star\})}{A!} iq\hat{\mathbf{x}}_k | \{p_\star\} \rangle_{Q_k},
 \end{aligned} \quad (214)$$

where $C \equiv \phi_m^*(0)\phi_n(0)$, and $Q_k = \langle \{p_\star'\} | e^{iq\hat{\mathbf{x}}_k} | \{p_\star\} \rangle$ is the matrix elements of the k th nucleon shift operator $e^{iq\hat{\mathbf{x}}_k}$ over the multiparticle state of free nucleons (diagonal in nuclear spin), whose explicit expression is given by (211). Therefore, after calculations similar to that above, f_{mn}^k takes the form

$$\begin{aligned}
 f_{mn}^k &= \int C \prod_{i,i \neq k}^A \frac{d\mathbf{p}_i^* 2E_{p_i^*}}{(2\pi)^3 2E_{p_i^*}} \\
 &\times \left(\frac{d\mathbf{p}_k^* d\mathbf{p}_k^* \sqrt{2E_{p_k^*} 2E_{p_k^*+q}}}{(2\pi)^3 \sqrt{2E_{p_k^*} 2E_{p_k^*}}} \delta^3(\mathbf{p}_k^* + \mathbf{q} - \mathbf{p}_k^*) \right) \\
 &\times \tilde{\Psi}_m^*(\{\mathbf{p}_\star^{(k)}\}) \tilde{\Psi}_n(\{\mathbf{p}_\star\}).
 \end{aligned}$$

After integration over $d\mathbf{p}_k^*$, normalization factors are cancelled out, and there occurs a “shift” of the argument of the wave function for the final nuclear state ($\mathbf{p}_k^* \rightarrow \mathbf{p}_k^* + \mathbf{q}$),

$$\begin{aligned}
 f_{mn}^k &= \int C \left(\prod_{i \neq k}^A \frac{d\mathbf{p}_i^*}{(2\pi)^3} \right) \left(\frac{d\mathbf{p}_k^* \sqrt{2E_{p_k^*+q} 2E_{p_k^*}}}{(2\pi)^3 \sqrt{2E_{p_k^*} 2E_{p_k^*+q}}} \right) \\
 &\times \tilde{\Psi}_m^*(\{\mathbf{p}_\star^{(k)}\}) \tilde{\Psi}_n(\{\mathbf{p}_\star\}) \\
 &= \int C \left(\prod_i^A \frac{d\mathbf{p}_i^*}{(2\pi)^3} \right) \tilde{\Psi}_m^*(\{\mathbf{p}_\star^{(k)}\}) \tilde{\Psi}_n(\{\mathbf{p}_\star\}).
 \end{aligned} \quad (215)$$

Now, if one takes $C \equiv (2\pi)^3 \delta^3(\sum_j^A \mathbf{p}_j^*)$, nuclear states (213) used above fully correspond to the definition in Eq. (197), and the nuclear matrix in the nuclear rest frame (78) does have the form ($\mathbf{p}$ -representation)

$$\begin{aligned}
 f_{mn}^k &= \langle m | e^{iq\hat{\mathbf{x}}_k} | n \rangle = \int \left(\prod_i^A \frac{d\mathbf{p}_i^*}{(2\pi)^3} \right) \\
 &\times \tilde{\Psi}_m^*(\{\mathbf{p}_\star^{(k)}\}) \tilde{\Psi}_n(\{\mathbf{p}_\star\}) (2\pi)^3 \delta^3 \left(\sum_{i=1}^A \mathbf{p}_i^* \right).
 \end{aligned} \quad (216)$$

Note that in the rest frame of the nucleus (where in the initial state there was $\sum_i^A \mathbf{p}_i^* = 0$ by definition), the total momentum of all nucleons in the final state is $\sum_i^A \mathbf{p}_i^* = \sum_{i \neq k}^A \mathbf{p}_i^* + \mathbf{p}_k^* \sum_{i \neq k}^A \mathbf{p}_i^* + \mathbf{p}_k^* + \mathbf{q} = \mathbf{q}$, i.e., this total momentum is equal to the momentum $\mathbf{q}$ transferred by the neutrino to the nucleus. It seems as if the matrix element $\langle m | e^{iq\hat{\mathbf{x}}_k} | n \rangle$ corresponds to the transition of the nucleus from the initial state at rest $|n\rangle$ to the final state $\langle m |$ that moves in the laboratory coordinate system with the momentum $\mathbf{q}$. The nucleus does move after interaction with the momentum $\mathbf{q}$ in the lab system. But the latter no longer coincides with the nuclear rest frame where the internal state of the nucleus is described by the function $\langle m |$.

10.2.3. Hadronic structure f_{mn}^k . Auxiliary calculations. Let us verify the relation between the $\mathbf{p}$ - and $\mathbf{x}$ -representations of the matrix element $f_{mn}^k = \langle m | e^{iq\hat{\mathbf{x}}_k} | n \rangle$. To this end, we use formulas from appendix 10.1 (section 10.1.3)

$$\begin{aligned}
 \langle \{x\} | \{p\} \rangle &= A! \prod_j^A u_{m_j}(\mathbf{p}_j, s_j) e^{i\mathbf{p}_j \mathbf{x}_j}, \\
 \langle \{p\} | \{x\} \rangle &= A! \prod_j^A u_{m_j}^\dagger(\mathbf{p}_j, s_j) e^{-i\mathbf{p}_j \mathbf{x}_j}; \\
 d\tilde{\mathbf{p}}_i &\equiv \frac{d\mathbf{p}_i}{(2\pi)^3 \sqrt{2E_{p_i}}},
 \end{aligned}$$

and the definition of the wave functions for the multiparticle system in the (coordinate) $\mathbf{x}$ -representation (182), (183), and (185) in terms of the functions in the $\mathbf{p}$ -representation. If one uses expressions in the $\mathbf{p}$ -representation in the rest frame of the nucleus, i.e., in the $\mathbf{p}^*$ -representation, one should consider the difference in normalizations of these momentum representations for the wave functions by introducing a factor $\sqrt{C}$. Then (185) can be written as

$$\begin{aligned}
 |n\rangle_x &\equiv \Psi_n(\{x\}) = \sqrt{C} \int \prod_i^A d\tilde{\mathbf{p}}_i^* \frac{\tilde{\Psi}_n(\{p_\star\})}{A!} \\
 &\times \langle \{x\} | \{p_\star\} \rangle = \sqrt{C} \int \prod_i^A d\tilde{\mathbf{p}}_i^* u_{m_i}(\mathbf{p}_i^*, s_i) \\
 &\times e^{i\mathbf{p}_i^* \mathbf{x}_i} \tilde{\Psi}_n(\{p_\star\}), \\
 \langle m |_x &\equiv \Psi_m^*(\{x\}) \\
 &= \sqrt{C} \int \prod_j^A d\tilde{\mathbf{p}}_j^* \frac{\tilde{\Psi}_m^*(\{p_\star\})}{A!} \langle \{p_\star\} | \{x\} \rangle \\
 &= \sqrt{C} \int \prod_j^A d\tilde{\mathbf{p}}_j^* u_{m_j}^\dagger(\mathbf{p}_j^*, s_j) e^{-i\mathbf{p}_j^* \mathbf{x}_j} \tilde{\Psi}_m^*(\{p_\star\}).
 \end{aligned} \quad (217)$$

Using these wave functions $\langle m |_x$ and $| n \rangle_x$, we calculate the matrix element

$$f_{mn}^k = \int \left(\prod_{i=1}^A d\mathbf{x}_i \right) \psi_m^*(\mathbf{x}_1, \dots, \mathbf{x}_A) e^{iq\mathbf{x}_k} \psi_n(\mathbf{x}_1, \dots, \mathbf{x}_A) \\ = \langle m | e^{iq\hat{\mathbf{x}}_k} | n \rangle_x$$

in the rest frame of the nucleus. After canceling all spin indices (since the shift operator does not depend on them), we have

$$f_{mn}^k = \langle m | e^{iq\hat{\mathbf{x}}_k} | n \rangle_x = \int \left(\prod_{i=1}^A d\mathbf{x}_i \right) \psi_m^* \\ \times (\{x\}) e^{iq\hat{\mathbf{x}}_k} \psi_n(\{x\}) = \int \left(\prod_{i=1}^A d\mathbf{x}_i \right) \\ \times \left(\prod_{j,l} \frac{d\mathbf{p}_j^* d\mathbf{p}_l^*}{(2\pi)^6 \sqrt{2E_{\mathbf{p}_j^*} 2E_{\mathbf{p}_l^*}}} \right) \frac{C \tilde{\psi}_m^*(\{p^*\}_j) \tilde{\psi}_n(\{p^*\}_l)}{(A!)^2} \\ \times \langle \{p^*\}_j | \{x\}_j \rangle e^{iq\hat{\mathbf{x}}_k} \langle \{x\}_l | \{p^*\}_l \rangle.$$

Since, in this case, the operator $e^{iq\hat{\mathbf{x}}_k}$ acts on the momentum variables, integration over variables $\mathbf{x}_i$ actually gives rise to the multiparticle unit vector in the $\mathbf{x}$ -representation (179), $\hat{I}_{x,A} = \int \prod_{i=1}^A d\mathbf{x}_i \frac{|\{x\}_i\rangle \langle \{x\}_i|}{A!}$. Changing, for convenience, designations of the integration variables and removing this unit operator, we obtain

$$f_{mn}^k = \int \prod_{j,l} \frac{d\mathbf{p}_j^* d\mathbf{p}_l^*}{(2\pi)^6 \sqrt{2E_{\mathbf{p}_j^*} 2E_{\mathbf{p}_l^*}}} \\ \times \frac{C \tilde{\psi}_m^*(\{p^*\}_j) \tilde{\psi}_n(\{p^*\}_l)}{A!} \underbrace{\langle \{p^*\}_j | e^{iq\hat{\mathbf{x}}_k} | \{p^*\}_l \rangle}_{Q_k}.$$

This expression for f_{mn}^k exactly coincides with formula (214), which involves the matrix element of k th nucleon shift operator $e^{iq\hat{\mathbf{x}}_k}$ over the multiparticle state of free nucleons Q_k (211). Therefore, considering $C = (2\pi)^3 \delta^3(\sum_j \mathbf{p}_j^*)$, one obtains Eq. (216)

$$f_{mn}^k = \int \left(\prod_l \frac{d\mathbf{p}_l^*}{(2\pi)^3} \right) \tilde{\psi}_m^*(\{p^*\}, \mathbf{p}_k^* \\ \rightarrow \mathbf{p}_k^* + \mathbf{q}) \tilde{\psi}_n(\{p^*\}) (2\pi)^3 \delta^3 \left(\sum_i \mathbf{p}_i^* \right).$$

Thus, with the $\mathbf{x}$ -representation for the initial and final states (the first formulas in (217)) as a starting point, the formula for $f_{mn}^k(\mathbf{q})$ in the $\mathbf{p}$ -representation was obtained.

Let us consider an opposite case, i.e., we obtain an expression for f_{mn}^k in the $\mathbf{x}$ -representation with the expression for f_{mn}^k in the $\mathbf{p}$ -representation as a starting point.

By analogy with the arguments given for Eqs. (217), we use formulas from expressions for wave functions of multiparticle systems (182), (183), and (185) for the initial and final states of the nucleus in its center-of-mass system, i.e., in the $\mathbf{p}^*$ -representation, by considering the difference in the $\mathbf{p}$ -representation and $\mathbf{p}^*$ -representation normalizations

$$\tilde{\psi}_n(p_\star) = \frac{1}{\sqrt{C}} \int \left(\prod_i \frac{d\mathbf{y}_i}{\sqrt{2E_{\mathbf{p}_i^*}}} \right) \frac{\psi_n(\{y\})}{A!} \langle \{p_\star\} | \{y\} \rangle \\ = \frac{1}{\sqrt{C}} \int \left(\prod_i d\mathbf{y}_i \frac{u_i^+(\mathbf{p}_i^*, s_i)}{\sqrt{2E_{\mathbf{p}_i^*}}} e^{-i\mathbf{p}_i^* \cdot \mathbf{y}_i} \right) \psi_n(\{y\}), \quad (218)$$

$$\tilde{\psi}_m^*(p_\star) = \frac{1}{\sqrt{C}} \int \left(\prod_i \frac{d\mathbf{x}_i}{\sqrt{2E_{\mathbf{p}_i^*}}} \right) \frac{\psi_m^*(\{x\})}{A!} \langle \{p_\star\} | \{x\} \rangle^* \\ = \frac{1}{\sqrt{C}} \int \left(\prod_i d\mathbf{x}_i \frac{u_i^-(\mathbf{p}_i^*, s_i)}{\sqrt{2E_{\mathbf{p}_i^*}}} e^{i\mathbf{p}_i^* \cdot \mathbf{x}_i} \right) \psi_m^*(\{x\}).$$

Then, according to the first formula on the right in (218), for the “shifted” final state wave function involved in expression (216) for f_{mn}^k one can write

$$\tilde{\psi}_m^*(\{p^*\}, \mathbf{p}_k^* \rightarrow \mathbf{p}_k^* + \mathbf{q}) = \int \left(\prod_{i \neq k} \frac{d\mathbf{z}_i}{\sqrt{2E_{\mathbf{p}_i^*}}} \right) \\ \times \frac{d\mathbf{z}_k}{\sqrt{2E_{\mathbf{p}_k^* + \mathbf{q}}}} \frac{\psi_m^*(\{z\})}{\sqrt{C} A!} \langle \{z\} | \{p^*\}, \mathbf{p}_k^* \rightarrow \mathbf{p}_k^* + \mathbf{q} \rangle. \quad (219)$$

Let us take it that the “shifted” scalar product $\langle \{z\} | \{p^*\}, \mathbf{p}_k^* \rightarrow \mathbf{p}_k^* + \mathbf{q} \rangle$ can be written as

$$\langle \dots, z_k, \dots | \dots, \mathbf{p}_{k-1}^*, \mathbf{p}_k^* \rightarrow \mathbf{p}_k^* + \mathbf{q}, \mathbf{p}_{k+1}^*, \dots \rangle \\ = \langle \dots, z_{k-1}, z_{k+1}, \dots | \dots, \mathbf{p}_{k-1}^*, \mathbf{p}_{k+1}^*, \dots \rangle \langle z_k | \mathbf{p}_k^* + \mathbf{q} \rangle,$$

then, according to (184), the following expression is valid for it:

$$\langle \{z\} | \{p^*\}, \mathbf{p}_k^* \rightarrow \mathbf{p}_k^* + \mathbf{q} \rangle \\ = (A-1)! \prod_{j \neq k}^{A-1} u_j(\mathbf{p}_j^*, s_j) e^{i\mathbf{p}_j^* \cdot \mathbf{z}_j} \langle z_k | \mathbf{p}_k^* + \mathbf{q} \rangle.$$

Considering (208), i.e.,

$$| \mathbf{p}_k^* + \mathbf{q} \rangle = e^{iq\mathbf{x}_k} \frac{\sqrt{2E_{\mathbf{p}_k^* + \mathbf{q}}}}{\sqrt{2E_{\mathbf{p}_k^*}}} | \mathbf{p}_k^* \rangle, \quad \text{and, in addition,} \\ \langle z_k | e^{iq\mathbf{x}_k} = \langle z_k | e^{iq\mathbf{z}_k}, \quad \text{we have}$$

$$\begin{aligned}
\langle z_k | \mathbf{p}_k^\star + \mathbf{q} \rangle &= \frac{\sqrt{2E_{\mathbf{p}_k^\star + \mathbf{q}}}}{\sqrt{2E_{\mathbf{p}_k^\star}}} \langle z_k | e^{iqz_k} | \mathbf{p}_k^\star \rangle \\
&= \frac{\sqrt{2E_{\mathbf{p}_k^\star + \mathbf{q}}}}{\sqrt{2E_{\mathbf{p}_k^\star}}} e^{iqz_k} \langle z_k | \mathbf{p}_k^\star \rangle \\
&= \frac{\sqrt{2E_{\mathbf{p}_k^\star + \mathbf{q}}}}{\sqrt{2E_{\mathbf{p}_k^\star}}} e^{iqz_k} u_k(\mathbf{p}_k^\star, s_k) e^{iz_k p_k^\star}.
\end{aligned} \quad (220)$$

Then, in view of possible A permutations of the k th nucleon for the “shifted” final state wave function (219), the following expression is obtained in the coordinate representation:

$$\begin{aligned}
\tilde{\Psi}_m^*(\{p_\star\}, \mathbf{p}_k^\star \rightarrow \mathbf{p}_k^\star + \mathbf{q}) &= \int \left(\prod_{i \neq k} \frac{dz_i}{\sqrt{2E_{\mathbf{p}_i^\star}}} \right) \\
&\times \frac{dz_k e^{iqz_k}}{\sqrt{2E_{\mathbf{p}_k^\star + \mathbf{q}}}} \frac{\Psi_n^*(\{z\})}{A! \sqrt{C}} \frac{\sqrt{2E_{\mathbf{p}_k^\star + \mathbf{q}}}}{\sqrt{2E_{\mathbf{p}_k^\star}}} A! \prod_j u_j(\mathbf{p}_j^\star, s_j) e^{ip_j^\star z_j} \\
&= \frac{1}{\sqrt{C}} \int \left(\prod_i \frac{dz_i}{\sqrt{2E_{\mathbf{p}_i^\star}}} \frac{u_i(\mathbf{p}_i^\star, s_i)}{\sqrt{2E_{\mathbf{p}_i^\star}}} e^{ip_i^\star z_i} \right) e^{iqz_k} \Psi_n^*(\{z\}).
\end{aligned} \quad (221)$$

Shifted wave function (219) can also be written in terms of the spinor functions $u_i(\mathbf{p}_i^\star, r_i)$, that is, with the use of the second (or right) formula from (218), as

$$\begin{aligned}
\tilde{\Psi}_m^*(\{p_\star\}, \mathbf{p}_k^\star \rightarrow \mathbf{p}_k^\star + \mathbf{q}) &= \frac{1}{\sqrt{C}} \int \left(\prod_{i \neq k} \frac{dz_i}{\sqrt{2E_{\mathbf{p}_i^\star}}} \frac{u_i(\mathbf{p}_i^\star, r_i)}{\sqrt{2E_{\mathbf{p}_i^\star}}} e^{ip_i^\star z_i} \right) \\
&\times \left(dz_k \frac{u_k(\mathbf{p}_k^\star + \mathbf{q}, r_k)}{\sqrt{2E_{\mathbf{p}_k^\star + \mathbf{q}}}} e^{i(\mathbf{p}_k^\star + \mathbf{q})z_k} \right) \Psi_m^*(\{z\}).
\end{aligned}$$

Further, one can use the definition of the scalar product (172) $\langle z | \mathbf{p}^\star \rangle = u(\mathbf{p}^\star, s_k) e^{ip^\star z}$ in the form $\langle z_k | \mathbf{p}_k^\star + \mathbf{q} \rangle = u(\mathbf{p}_k^\star + \mathbf{q}, s_k) e^{i(\mathbf{p}_k^\star + \mathbf{q})z_k}$, condition (208), and the action of the shift operator on the z -state $\langle z_k | e^{iqz_k} = e^{iqz_k} \langle z_k |$. And again, one obtains expression (220)

$$\begin{aligned}
u_k(\mathbf{p}_k^\star + \mathbf{q}, s_k) e^{i(\mathbf{p}_k^\star + \mathbf{q})z_k} &= \frac{\sqrt{2E_{\mathbf{p}_k^\star + \mathbf{q}}}}{\sqrt{2E_{\mathbf{p}_k^\star}}} e^{iqz_k} \langle z_k | \mathbf{p}_k^\star \rangle \\
&= e^{iqz_k} \frac{\sqrt{2E_{\mathbf{p}_k^\star + \mathbf{q}}}}{\sqrt{2E_{\mathbf{p}_k^\star}}} u_k(\mathbf{p}_k^\star, s_k) e^{ip_k^\star z_k}.
\end{aligned}$$

As a result, shifted wave function (219) takes the form coinciding with (221)

$$\begin{aligned}
\tilde{\Psi}_m^*(\{p_\star\}, \mathbf{p}_k^\star \rightarrow \mathbf{p}_k^\star + \mathbf{q}) &= \frac{1}{\sqrt{C}} \int \left(\prod_i \frac{dz_i}{\sqrt{2E_{\mathbf{p}_i^\star}}} \frac{u_i(\mathbf{p}_i^\star, r_i)}{\sqrt{2E_{\mathbf{p}_i^\star}}} e^{ip_i^\star z_i} \right) e^{iqz_k} \Psi_m^*(\{z\}).
\end{aligned} \quad (222)$$

Using (222), we obtain expressions for $f_{mn}^k(\mathbf{q})$ given by Eq. (216)

$$\begin{aligned}
f_{mn}^k(\mathbf{q}) &= \int \left(\prod_i \frac{d\mathbf{p}_i^\star}{(2\pi)^3} \right) \tilde{\Psi}_m^*(\{p_\star\}, \mathbf{p}_k^\star \\
&\rightarrow \mathbf{p}_k^\star + \mathbf{q}) \tilde{\Psi}_n(\{p_\star\}) (2\pi)^3 \delta^3 \left(\sum_{i=1}^A \mathbf{p}_i^\star \right).
\end{aligned}$$

Since $\tilde{\Psi}_m^*(\{p_\star\})$ and $\tilde{\Psi}_n(\{p_\star\})$ are scalar functions, they can be permuted, i.e., $f_{mn}^k(\mathbf{q})$ can be written as follows:

$$\begin{aligned}
f_{mn}^k(\mathbf{q}) &= \int \left(\prod_i \frac{d\mathbf{p}_i^\star}{(2\pi)^3} \right) \tilde{\Psi}_n(\{p_\star\}) \tilde{\Psi}_m^* \\
&\times (\{p_\star\}, \mathbf{p}_k^\star \rightarrow \mathbf{p}_k^\star + \mathbf{q}) (2\pi)^3 \delta^3 \left(\sum_{i=1}^A \mathbf{p}_i^\star \right).
\end{aligned}$$

Substituting (218) and (222) into this expression, we have

$$\begin{aligned}
f_{mn}^k(\mathbf{q}) &= \int \left(\prod_i \frac{d\mathbf{p}_i^\star}{(2\pi)^3} \right) \frac{(2\pi)^3 \delta^3 \left(\sum_{i=1}^A \mathbf{p}_i^\star \right)}{C} \\
&\times \int \left(\prod_l \frac{dy_l}{\sqrt{2E_{\mathbf{p}_l^\star}}} \frac{u_l^\dagger(\mathbf{p}_l^\star, s_l)}{\sqrt{2E_{\mathbf{p}_l^\star}}} e^{-ip_l^\star y_l} \right) \Psi_n(\{y\}) \\
&\times \int \left(\prod_i \frac{dz_i}{\sqrt{2E_{\mathbf{p}_i^\star}}} \frac{u_i(\mathbf{p}_i^\star, r_i)}{\sqrt{2E_{\mathbf{p}_i^\star}}} e^{ip_i^\star z_i} \right) e^{iqz_k} \Psi_m^*(\{z\}).
\end{aligned}$$

If, as previously, the normalization coefficient is $C = (2\pi)^3 \delta^3 \left(\sum_{i=1}^A \mathbf{p}_i^\star \right)$ and by the normalization condition for Dirac spinors there is

$$\frac{u_l^\dagger(\mathbf{p}_l^\star, s_l)}{\sqrt{2E_{\mathbf{p}_l^\star}}} \frac{u_i(\mathbf{p}_i^\star, r_i)}{\sqrt{2E_{\mathbf{p}_i^\star}}} = \delta_{li} \delta_{s_l, r_i},$$

$f_{mn}^k(\mathbf{q})$ takes the form of the multiple integral

$$\begin{aligned}
&\int \left(\prod_i \frac{d\mathbf{p}_i^\star}{(2\pi)^3} \right) \int \left(\prod_{i,l} dy_l dz_i \delta_{li} \delta_{s_l, r_i} e^{ip_i^\star (z_i - y_l)} \right) \\
&\times \Psi_n(\{y\}) e^{iqz_k} \Psi_m^*(\{z\}),
\end{aligned}$$

or

$$\begin{aligned}
f_{mn}^k(\mathbf{q}) &= \int \left(\prod_l \int dy_l dz_l \right) \\
&\times \left[\prod_i \int \left(\frac{d\mathbf{p}_i^\star}{(2\pi)^3} \right) e^{ip_i^\star (z_i - y_i)} \right] \\
&\times \Psi_n(\{y\}) e^{iqz_k} \Psi_m^*(\{z\}),
\end{aligned}$$

where A -tuple integration over $d\mathbf{p}_i^\star / (2\pi)^3$ yields A $\delta^3(\mathbf{z}_l - \mathbf{y}_l)$, which remove A integrations over $d\mathbf{y}_l$, which results in the ultimate formula for $f_{mn}^k(\mathbf{q})$ in the coordinate representation

$$\begin{aligned} f_{mn}^k(\mathbf{q}) &= \langle m | e^{i\mathbf{q}\hat{\mathbf{x}}_k} | n \rangle_x \\ &= \int \prod_i^A d\mathbf{z}_i \psi_m^*(\{\mathbf{z}\}) e^{i\mathbf{q}\mathbf{z}_k} \psi_n(\{\mathbf{z}\}) \\ &\equiv \langle m | e^{i\mathbf{q}\mathbf{x}_k} | n \rangle = \int \prod_{i=1}^A d\mathbf{x}_i \psi_m^*(\mathbf{x}_1, \dots, \mathbf{x}_A) \\ &\quad \times e^{i\mathbf{q}\mathbf{x}_k} \psi_n(\mathbf{x}_1, \dots, \mathbf{x}_A). \end{aligned} \quad (223)$$

It is seen that if one puts $\mathbf{q} = 0$ in (223), it should pass into the normalization condition for states $\langle m | n \rangle$ in the coordinate representation, that is, into expression (187) written in the form $\langle m | n \rangle_x = \delta_{mn}$. Direct verification of fulfillment of this condition will mean correct use of the transition (from $\mathbf{p}^\star$ - to $\mathbf{p}$ -representation) normalization factor $C = (2\pi)^3 \delta^3(\sum_{i=1}^A \mathbf{p}_i^\star)$.

Indeed, let us derive expressions for the normalization condition for nuclear states in the rest frame of the nucleus (196) in terms of wave functions in the coordinate representation, beginning with formula (196) given in the momentum representation (in the nuclear rest frame)

$$\begin{aligned} \langle m | n \rangle &= \int \left(\prod_i^A \frac{d\mathbf{p}_i^\star}{(2\pi)^3} \right) \tilde{\psi}_n(\{\mathbf{p}_\star\}) \tilde{\psi}_m^*(\{\mathbf{p}_\star\}) \\ &\quad \times (2\pi)^3 \delta^3\left(\sum_{i=1}^A \mathbf{p}_i^\star\right) = \delta_{mn}. \end{aligned}$$

To this end, the product $\tilde{\psi}_n(\{\mathbf{p}_\star\}) \tilde{\psi}_m^*(\{\mathbf{p}_\star\})$ should be expressed in terms of x -variables, i.e., one should use second (or rightmost) formulas from relations (218)

$$\begin{aligned} \tilde{\psi}_n(\{\mathbf{p}_\star\}) \tilde{\psi}_m^*(\{\mathbf{p}_\star\}) &= \frac{1}{C} \int \left(\prod_i^A d\mathbf{y}_i \frac{u_i^\dagger(\mathbf{p}_i^\star, s_i)}{\sqrt{2E_{\mathbf{p}_i^\star}}} e^{-i\mathbf{p}_i^\star \mathbf{y}_i} \right) \\ &\quad \times \psi_n(\{\mathbf{y}\}) \int \left(\prod_j^A d\mathbf{z}_j \frac{u_j(\mathbf{p}_j^\star, s_j)}{\sqrt{2E_{\mathbf{p}_j^\star}}} e^{i\mathbf{p}_j^\star \mathbf{z}_j} \right) \psi_m^*(\{\mathbf{z}\}) \\ &= \int \left(\prod_{i,j}^A d\mathbf{y}_i d\mathbf{z}_j \frac{u_i^\dagger(\mathbf{p}_i^\star, s_i) u_j(\mathbf{p}_j^\star, s_j)}{\sqrt{2E_{\mathbf{p}_i^\star}} \sqrt{2E_{\mathbf{p}_j^\star}}} e^{i\mathbf{p}_j^\star \mathbf{z}_j - i\mathbf{p}_i^\star \mathbf{y}_i} \right) \\ &\quad \times \psi_n(\{\mathbf{y}\}) \psi_m^*(\{\mathbf{z}\}). \end{aligned}$$

Then $\langle m | n \rangle_x$, in view of $C = (2\pi)^3 \delta^3(\sum_{i=1}^A \mathbf{p}_i^\star)$, takes the form

$$\begin{aligned} &\int \left(\prod_i^A \frac{d\mathbf{p}_i^\star}{(2\pi)^3} \right) \left(\prod_{i,j}^A d\mathbf{y}_i d\mathbf{z}_j \frac{u_i^\dagger(\mathbf{p}_i^\star, s_i) u_j(\mathbf{p}_j^\star, s_j)}{\sqrt{2E_{\mathbf{p}_i^\star}} \sqrt{2E_{\mathbf{p}_j^\star}}} e^{i\mathbf{p}_j^\star \mathbf{z}_j - i\mathbf{p}_i^\star \mathbf{y}_i} \right) \\ &\quad \times \psi_n(\{\mathbf{y}\}) \psi_m^*(\{\mathbf{z}\}) \end{aligned}$$

Considering normalization of spinors $u_i^\dagger(\mathbf{p}_i^\star, s_i) u_j(\mathbf{p}_j^\star, s_j) = 2E_{\mathbf{p}_i^\star} \delta^3(\mathbf{p}_i^\star - \mathbf{p}_j^\star) \delta_{s_i s_j} \delta_{ij}$, we have

$$\begin{aligned} &\int \left(\prod_i^A \frac{d\mathbf{p}_i^\star}{(2\pi)^3} \right) \\ &\quad \times \left(\prod_{i,j}^A d\mathbf{y}_i d\mathbf{z}_j \frac{E_{\mathbf{p}_i^\star} \delta^3(\mathbf{p}_i^\star - \mathbf{p}_j^\star) \delta_{s_i s_j} \delta_{ij}}{\sqrt{E_{\mathbf{p}_i^\star}} \sqrt{E_{\mathbf{p}_j^\star}}} e^{i\mathbf{p}_i^\star (\mathbf{z}_j - \mathbf{y}_i)} \right) \\ &\quad \times \psi_n(\{\mathbf{y}\}) \psi_m^*(\{\mathbf{z}\}). \end{aligned}$$

Given that $\int \frac{d\mathbf{p}_i^\star}{(2\pi)^3} e^{i\mathbf{p}_i^\star (\mathbf{z}_j - \mathbf{y}_i)} \delta^3(\mathbf{z}_j - \mathbf{y}_i)$, one can integrate the expression

$$\begin{aligned} &\left(\prod_i^A \int \frac{d\mathbf{p}_i^\star}{(2\pi)^3} e^{i\mathbf{p}_i^\star (\mathbf{z}_j - \mathbf{y}_i)} \right) \\ &\quad \times \int \left(\prod_{i,j}^A d\mathbf{y}_i d\mathbf{z}_j \right) \psi_n(\{\mathbf{y}\}) \psi_m^*(\{\mathbf{z}\}), \end{aligned}$$

over $\mathbf{p}_i^\star$ and obtain $\int \left(\prod_{i,j}^A d\mathbf{y}_i d\mathbf{z}_j \delta^3(\mathbf{z}_j - \mathbf{y}_i) \right) \times \psi_n(\{\mathbf{y}\}) \psi_m^*(\{\mathbf{z}\})$, whence one ultimately gets

$$\langle m | n \rangle_x = \int \left(\prod_i^A d\mathbf{y}_i \right) \psi_n(\{\mathbf{y}\}) \psi_m^*(\{\mathbf{y}\}) = \delta_{mn},$$

which coincides with Eq (187) at $m = n$.

10.3. Transformation of the Cross Section for $\nu A \rightarrow \nu A^{(*)}$ Scattering Using Delta Functions

In the case of the $2 \rightarrow 2$ scattering like $\nu(p_1) + A(p_2) \rightarrow \nu(p_1') + A^{(*)}(p_2')$, after integration over the angle $d\phi$ and over $d\mathbf{p}_2'$ using the delta function, the equation for the cross section is obtained,

$$\begin{aligned} d\sigma &\equiv \frac{dw E_1 E_2}{n_1 n_2 I} \\ &= -2\pi \frac{\delta(E_1 + E_2 - E_1' - E_2') |\mathcal{M}|^2 |\mathbf{p}_1'|^2 d|\mathbf{p}_1'| d\cos\theta}{64\pi^2 I E_1' E_2'}, \end{aligned}$$

from which, for a zero-mass (anti)neutrino ($|\mathbf{p}_1'| = E_1'$ and $I = E_1 E_2$), follows the formula for the differential cross section of the (anti)neutrino–nucleus scattering

$$\begin{aligned} \frac{d\sigma}{dE'_1 d\cos\theta} &= -\frac{\delta(E_1 + E_2 - E'_1 - E'_2) |\mathcal{M}|^2 (E'_1)^2}{32\pi(E_1 E_2) E'_1 E'_2} \\ &= -\frac{E'_1 |\mathcal{M}|^2 \delta(E_1 + E_2 - E'_1 - E'_2(E'_1))}{E_1 32\pi E_2 E'_2}. \end{aligned}$$

Considering that in the laboratory coordinate system the nucleus before interaction is supposed to be at rest and in an internal quantum state n , for the kinematic variables in this formula we have

$$E_1 = E_\nu, \quad E'_1 = E'_n, \quad E_2 = E_A = m_A + \varepsilon_n,$$

$$E'_2 = E'_A = m_A + \varepsilon_m + T_A,$$

$$E_2 - E'_2 = \varepsilon_n - \varepsilon_m - T_A.$$

At $\Delta\varepsilon_{mn} = \varepsilon_m - \varepsilon_n$ we obtain

$$\begin{aligned} \frac{d^2\sigma_{mn}}{dE'_\nu d\cos\theta} &= -\frac{E'_\nu |i\mathcal{M}_{mn}|^2}{2^5 \pi E_\nu (m_A + \varepsilon_n)} \\ &\times \frac{\delta(E_\nu - E'_\nu - T_A(E'_\nu) - \Delta\varepsilon_{mn})}{m_A + T_A + \varepsilon_m}, \end{aligned} \quad (224)$$

where E'_ν and $\cos\theta$ are two independent variables characterizing the entire scattering process. Since the nuclear recoil kinetic energy

$$\begin{aligned} T_A &= \sqrt{m_A^2 + \mathbf{q}^2} - m_A \\ &= \sqrt{m_A^2 + E_\nu^2 + (E'_\nu)^2 - 2E'_\nu E_\nu \cos\theta} - m_A \end{aligned} \quad (225)$$

depends on the neutrino energy E'_ν , integration of the delta function over the variable E'_ν in (224) should be performed using the formula

$$\delta(f(E'_\nu)) = \frac{\delta(E'_\nu - \tilde{E}'_\nu)}{|df(\tilde{E}'_\nu)/dE'_\nu|} = \frac{\delta(E'_\nu - \tilde{E}'_\nu)}{|-1 - dT_A(\tilde{E}'_\nu)/dE'_\nu|}. \quad (226)$$

Here

$$E'_\nu = \frac{E_\nu(m_A - \Delta\varepsilon_{mn}) + \Delta\varepsilon_{mn}^2/2 - m_A \Delta\varepsilon_{mn}}{m_A - \Delta\varepsilon_{mn} + E_\nu(1 - \cos\theta)} \quad (227)$$

is the solution of the equation

$$f(E'_\nu) \equiv E_\nu - \tilde{E}'_\nu - T_A(E'_\nu) - \Delta\varepsilon_{mn} = 0, \quad (228)$$

that reduces the argument of this delta function to zero and thus ensures its nonzero value²⁴. The expansion

$$\begin{aligned} (E_\nu - E'_\nu - \Delta\varepsilon_{mn} + m_A)^2 &= m_A^2 + E_\nu^2 + (E'_\nu)^2 + \Delta\varepsilon_{mn}^2 - 2E_\nu E'_\nu \\ &- 2E_\nu \Delta\varepsilon_{mn} + 2E_\nu m_A + 2E'_\nu \Delta\varepsilon_{mn} - 2E'_\nu m_A - 2m_A \Delta\varepsilon_{mn} \end{aligned} \quad (229)$$

is used. Solution (227) gives the dependence of the scattered neutrino energy on the scattering angle, which follows from the energy conservation condition in the form (228).

For the derivative of the nuclear kinetic energy with respect to the scattered neutrino energy (taken at the point E'_ν) there is an expression

$$\frac{dT_A(E'_\nu)}{dE'_\nu} = \frac{E'_\nu - E_\nu \cos\theta}{\sqrt{m_A^2 + E_\nu^2 + (E'_\nu)^2 - 2E'_\nu E_\nu \cos\theta}} = \frac{E'_\nu - E_\nu \cos\theta}{T_A(E'_\nu) + m_A},$$

in which $\cos\theta$ is (still) supposed to be independent of E'_ν , and the root in the denominator is replaced with $T_A(E'_\nu) + m_A$ by virtue of the definition of the kinetic energy (225). Since $f(\tilde{E}'_\nu) = 0$, E'_ν can be replaced with $E_\nu - T_A(E'_\nu) - \Delta\varepsilon_{mn}$,

$$E'_\nu = E_\nu - T_A(E'_\nu) - \Delta\varepsilon_{mn}$$

and for the derivative $\left| \frac{df(E'_\nu)}{dE'_\nu} \right| = \frac{dT_A(E'_\nu)}{dE'_\nu} + 1$ from (226) one obtains

$$\left| \frac{df(E'_\nu)}{dE'_\nu} \right| = \frac{E_\nu - T_A(E'_\nu) - \Delta\varepsilon_{mn} - E_\nu \cos\theta + T_A(E'_\nu) + m_A}{T_A(E'_\nu) + m_A} = \frac{E_\nu(1 - \cos\theta) + m_A - \Delta\varepsilon_{mn}}{T_A(E'_\nu) + m_A}.$$

²⁴ Considering (60), from the relation $E_\nu - E'_\nu - \Delta\varepsilon_{mn} = \sqrt{m_A^2 + E_\nu^2 + (E'_\nu)^2 - 2E'_\nu E_\nu \cos\theta} - m_A$ we have $(E_\nu - E'_\nu - \Delta\varepsilon_{mn} + m_A)^2 = m_A^2 + E_\nu^2 + (E'_\nu)^2 - 2E'_\nu E_\nu \cos\theta$, from which, using expansion (229), we obtain $\Delta\varepsilon_{mn}^2 - 2E_\nu E'_\nu - 2E_\nu \Delta\varepsilon_{mn} + 2E_\nu m_A + 2E'_\nu \Delta\varepsilon_{mn} - 2E'_\nu m_A - 2m_A \Delta\varepsilon_{mn} = -2E'_\nu E_\nu \cos\theta$. Then from $\Delta\varepsilon_{mn}^2 - 2m_A \Delta\varepsilon_{mn} + 2E_\nu(m_A - \Delta\varepsilon_{mn}) = 2E'_\nu(E_\nu - E'_\nu \cos\theta - \Delta\varepsilon_{mn} + m_A)$ there follows formula (227).

In view of (227) and (228), for T_A (corresponding to E'_v) one has²⁵

$$T_A = \frac{E_v(E_v - \Delta\epsilon_{mn})(1 - \cos\theta) + \Delta^2\epsilon_{mn}/2}{E_v(1 - \cos\theta) + m_A - \Delta\epsilon_{mn}}. \quad (230)$$

As a result, cross section (224) can be integrated over E'_v using delta function (226)

$$\begin{aligned} \frac{d\sigma_{mn}}{d\cos\theta} &= \int dE'_v \frac{(-1)E'_v |i\mathcal{M}_{mn}|^2}{2^5 \pi E_v(m_A + \epsilon_n) m_A + T_A + \epsilon_m} \frac{1}{\left| \frac{df(E'_v)}{dE'_v} \right|} \frac{\delta(E'_v - E_v)}{E_v(1 - \cos\theta) + m_A - \Delta\epsilon_{mn}} \\ &= \int dE'_v \frac{(-1)E'_v |i\mathcal{M}_{mn}|^2}{2^5 \pi E_v(m_A + \epsilon_n) m_A + T_A + \epsilon_m} \frac{\delta(E'_v - E_v)}{E_v(1 - \cos\theta) + m_A - \Delta\epsilon_{mn}} \frac{T_A + m_A}{E_v(1 - \cos\theta) + m_A - \Delta\epsilon_{mn}}. \end{aligned}$$

The integration is easily performed with the help of the $\delta(E'_v - E_v)$ function, i.e., the differential cross section depending on the initial neutrino energy E_v and the neutrino scattering angle θ is

$$\frac{d\sigma_{mn}}{d\cos\theta} = -\frac{E'_v |i\mathcal{M}_{mn}|^2}{2^5 \pi E_v(m_A + \epsilon_n) m_A + T_A(E'_v) + \epsilon_m} \frac{1}{E_v(1 - \cos\theta) + m_A - \Delta\epsilon_{mn}} \frac{m_A + T_A(E'_v)}{E_v(1 - \cos\theta) + m_A - \Delta\epsilon_{mn}}. \quad (231)$$

Here T_A and E'_v are defined by (230) and (227) in terms of $\cos\theta$ (and also E_v , m_A , and $\Delta\epsilon_{mn}$). Next, in (231) it is necessary to go from $d\cos\theta$ to dT_A . To this end, we use exact expression (230) for the dependence of T_A on $\cos\theta$

$$T_A = \frac{E_v(E_v - \Delta\epsilon_{mn})(1 - \cos\theta) + \Delta^2\epsilon_{mn}/2}{E_v(1 - \cos\theta) + m_A - \Delta\epsilon_{mn}} \equiv \frac{A(1 - \cos\theta) + B}{E(1 - \cos\theta) + C},$$

where $A = E_v(E_v - \Delta\epsilon_{mn})$, $B = \Delta^2\epsilon_{mn}/2$, $C = m_A - \Delta\epsilon_{mn}$, and $E = E_v$, and thus the Jacobian of the transformation takes the form

$$\begin{aligned} \frac{dT_A}{d\cos\theta} &= \frac{BE - AC}{(E(1 - \cos\theta) + C)^2} E_v \frac{\Delta^2\epsilon_{mn}/2 - (E_v - \Delta\epsilon_{mn})(m_A - \Delta\epsilon_{mn})}{[E_v(1 - \cos\theta) + m_A - \Delta\epsilon_{mn}]^2} \\ &= -E_v \frac{E_v(m_A - \Delta\epsilon_{mn}) + \Delta\epsilon_{mn}^2/2 - m_A\Delta\epsilon_{mn}}{[E_v(1 - \cos\theta) + m_A - \Delta\epsilon_{mn}]^2}, \end{aligned} \quad (232)$$

where it is considered that $\Delta^2\epsilon_{mn}/2 - (E_v - \Delta\epsilon_{mn})(m_A - \Delta\epsilon_{mn}) \Delta^2\epsilon_{mn}/2 - E_v(m_A - \Delta\epsilon_{mn}) + \Delta\epsilon_{mn}m_A - \Delta\epsilon_{mn}^2 = -[E_v(m_A - \Delta\epsilon_{mn}) + \Delta\epsilon_{mn}^2/2 - m_A\Delta\epsilon_{mn}]$.

Substituting this Jacobian and the expression for E'_v into (231), we obtain

$$\begin{aligned} \frac{d\sigma_{mn}}{dT_A} &= \frac{d\sigma_{mn}}{d\cos\theta} \frac{d\cos\theta}{dT_A} = \frac{d\sigma_{mn}}{d\cos\theta} \frac{-1}{dT_A} \frac{[E_v(1 - \cos\theta) + m_A - \Delta\epsilon_{mn}]^2}{E_v E_v(m_A - \Delta\epsilon_{mn}) + \Delta\epsilon_{mn}^2/2 - m_A\Delta\epsilon_{mn}} \\ &= \frac{|i\mathcal{M}_{mn}|^2}{2^5 \pi E_v(m_A + \epsilon_n) (m_A + T_A + \epsilon_m)} \frac{1}{E_v(1 - \cos\theta) + m_A - \Delta\epsilon_{mn}} \frac{(m_A + T_A)}{E_v} \frac{1}{E_v} \frac{[E_v(1 - \cos\theta) + m_A - \Delta\epsilon_{mn}]^2}{E_v E_v(m_A - \Delta\epsilon_{mn}) + \Delta\epsilon_{mn}^2/2 - m_A\Delta\epsilon_{mn}} \\ &= \frac{|i\mathcal{M}_{mn}|^2}{2^5 \pi E_v^2(m_A + \epsilon_n) (m_A + T_A + \epsilon_m)} \frac{1}{E_v(1 - \cos\theta) + m_A - \Delta\epsilon_{mn}} \frac{(m_A + T_A)}{E_v(m_A - \Delta\epsilon_{mn}) + \Delta\epsilon_{mn}^2/2 - m_A\Delta\epsilon_{mn}} \\ &\quad \times \frac{[E_v(1 - \cos\theta) + m_A - \Delta\epsilon_{mn}]^2}{E_v(m_A - \Delta\epsilon_{mn}) + \Delta\epsilon_{mn}^2/2 - m_A\Delta\epsilon_{mn}}. \end{aligned}$$

²⁵ From (228) it follows that $T_A(\tilde{E}'_v) = E_v - E'_v - \Delta\epsilon_{mn}$, where $\tilde{E}'_v = \frac{E_v(m_A - \Delta\epsilon_{mn}) + \Delta\epsilon_{mn}^2/2 - m_A\Delta\epsilon_{mn}}{m_A - \Delta\epsilon_{mn} + E_v(1 - \cos\theta)}$. That is, $T_A(\tilde{E}'_v) = E_v - \Delta\epsilon_{mn} -$

$\frac{E_v(m_A - \Delta\epsilon_{mn}) + \Delta\epsilon_{mn}^2/2 - m_A\Delta\epsilon_{mn}}{m_A - \Delta\epsilon_{mn} + E_v(1 - \cos\theta)} = \frac{(E_v - \Delta\epsilon_{mn})(m_A - \Delta\epsilon_{mn} + E_v(1 - \cos\theta))}{m_A - \Delta\epsilon_{mn} + E_v(1 - \cos\theta)} - \frac{E_v(m_A - \Delta\epsilon_{mn}) + \Delta\epsilon_{mn}^2/2 - m_A\Delta\epsilon_{mn}}{m_A - \Delta\epsilon_{mn} + E_v(1 - \cos\theta)}$. Then, $T_A(\tilde{E}'_v)[m_A - \Delta\epsilon_{mn} + E_v(1 - \cos\theta)] = (E_v - \Delta\epsilon_{mn})[(m_A - \Delta\epsilon_{mn}) + E_v(1 - \cos\theta)] - [E_v(m_A - \Delta\epsilon_{mn}) + \Delta\epsilon_{mn}^2/2 - m_A\Delta\epsilon_{mn}] = (E_v - \Delta\epsilon_{mn})(m_A - \Delta\epsilon_{mn}) + (E_v - \Delta\epsilon_{mn})E_v(1 - \cos\theta) - E_v(m_A - \Delta\epsilon_{mn}) - \Delta\epsilon_{mn}^2/2 + m_A\Delta\epsilon_{mn} = E_v(1 - \cos\theta) + (E_v - \Delta\epsilon_{mn}) + E_v(m_A - \Delta\epsilon_{mn}) - E_v(m_A - \Delta\epsilon_{mn}) - m_A\Delta\epsilon_{mn} + \Delta\epsilon_{mn}^2 - \Delta\epsilon_{mn}^2/2 + m_A\Delta\epsilon_{mn} = E_v(1 - \cos\theta) + (E_v - \Delta\epsilon_{mn}) + \Delta\epsilon_{mn}^2 - \Delta\epsilon_{mn}^2/2 \rightarrow (230)$.

Further, after obvious cancellations

$$\frac{E_\nu(m_A - \Delta\epsilon_{mn}) + \Delta\epsilon_{mn}^2/2 - m_A\Delta\epsilon_{mn}}{(m_A - \Delta\epsilon_{mn} + E_\nu(1 - \cos\theta))^2} \frac{[E_\nu(1 - \cos\theta) + m_A - \Delta\epsilon_{mn}]^2}{E_\nu(m_A - \Delta\epsilon_{mn}) + \Delta\epsilon_{mn}^2/2 - m_A\Delta\epsilon_{mn}} = 1,$$

the cross section takes the (simple) form

$$\frac{d\sigma_{mn}}{dT_A} = \frac{|i\mathcal{M}_{mn}|^2}{2^5\pi E_\nu^2(m_A + \epsilon_n)(m_A + T_A + \epsilon_m)} \equiv \frac{|i\mathcal{M}_{mn}|^2}{2^5\pi E_\nu^2 m_A} \tilde{C}_{2,mn}(T_A), \quad (233)$$

where

$$\tilde{C}_{2,mn}(T_A) = \frac{m_A}{(m_A + \epsilon_n)(m_A + T_A + \epsilon_m)} \frac{(m_A + T_A)}{(m_A + \epsilon_n)(m_A + T_A + \epsilon_m)}$$

On the other hand, since the nuclear recoil kinetic energy

$$\equiv \frac{1}{\left[1 + \frac{\epsilon_n}{m_A}\right] \left[1 + \frac{T_A + \epsilon_m}{m_A}\right]} = O(1).$$

$$T_A = \sqrt{m_A^2 + \mathbf{q}^2} - m_A \\ = \sqrt{m_A^2 + E_\nu^2 + (E'_\nu)^2 - 2E'_\nu E_\nu \cos\theta} - m_A$$

This expression for the cross section is obtained by integration of the delta function over E'_ν in the initial formula for the cross section (224).

depends on both the neutrino energy E'_ν and $\cos\theta$, integration of the delta function in (224) can be first performed over $\cos\theta$ using the relation

$$\delta(f(\cos\theta)) = \frac{\delta(\cos\theta - \cos\theta_i)}{|df(\cos\theta_i)/d\cos\theta|}. \quad (234)$$

Here

$$\begin{aligned} \cos\theta_i &= 1 - \frac{(E_\nu - E'_\nu)(m_A - \Delta\epsilon_{mn}) - \Delta\epsilon_{mn}(m_A - \Delta\epsilon_{mn}/2)}{E'_\nu E_\nu} \\ &= 1 + \frac{(m_A - \Delta\epsilon_{mn})}{E_\nu} - \frac{(m_A - \Delta\epsilon_{mn})}{E'_\nu} + \frac{\Delta\epsilon_{mn}m_A - \Delta\epsilon_{mn}^2/2}{E'_\nu E_\nu} \end{aligned} \quad (235)$$

is the value of $\cos\theta$ that reduces to zero the argument of the delta function at an (still) arbitrary value of the second free variable E'_ν , i.e., the solution of the equation

$$f(\cos\theta) \equiv E_\nu - E'_\nu - T_A(\cos\theta) - \Delta\epsilon_{mn} = 0. \quad (236)$$

“Reversal” of Eq. (235) yields a well-known formula for the dependence of E'_ν on $\cos\theta$

$$E'_\nu = \frac{E_\nu(m_A - \Delta\epsilon_{mn}) - m_A\Delta\epsilon_{mn} + \Delta\epsilon_{mn}^2/2}{(m_A - \Delta\epsilon_{mn}) + E_\nu(1 - \cos\theta)}.$$

One can obtain solution (235) beginning with equality (236) while explicitly considering the formula for T_A , i.e.,²⁶

$$\begin{aligned} E_\nu - E'_\nu + m_A - \Delta\epsilon_{mn} \\ = \sqrt{m_A^2 + E_\nu^2 + (E'_\nu)^2 - 2E'_\nu E_\nu \cos\theta}. \end{aligned}$$

From (235) it follows for derivatives that

$$\frac{d\cos\theta}{dE'_\nu} = \frac{E_\nu(m_A - \Delta\epsilon_{mn}) - \Delta\epsilon_{mn}m_A + \Delta\epsilon_{mn}^2/2}{(E'_\nu)^2 E_\nu}, \text{ or}$$

²⁶Indeed, raising both sides to the second power and using expansion (229), we have

$$\begin{aligned} \cos\theta &= \frac{m_A^2 + E_\nu^2 + (E'_\nu)^2 - (E_\nu - E'_\nu + m_A - \Delta\epsilon_{mn})^2}{2E'_\nu E_\nu} \\ &= \frac{-\Delta\epsilon_{mn}^2 + 2E_\nu E'_\nu - 2E_\nu m_A + 2E_\nu \Delta\epsilon_{mn} + 2E'_\nu m_A - 2E'_\nu \Delta\epsilon_{mn} + 2m_A \Delta\epsilon_{mn}}{2E'_\nu E_\nu} \\ &= 1 + \frac{E'_\nu(m_A - \Delta\epsilon_{mn}) - E_\nu(m_A - \Delta\epsilon_{mn}) + \Delta\epsilon_{mn}(m_A - \Delta\epsilon_{mn}/2)}{E'_\nu E_\nu} \\ &= 1 + \frac{(m_A - \Delta\epsilon_{mn})}{E_\nu} + \frac{-E_\nu(m_A - \Delta\epsilon_{mn}) + \Delta\epsilon_{mn}(m_A - \Delta\epsilon_{mn}/2)}{E'_\nu E_\nu}. \end{aligned}$$

$$\begin{aligned} \frac{dE'_v}{d\cos\theta} &= \frac{(E'_v)^2 E_v}{E_v(m_A - \Delta\epsilon_{mn}) - \Delta\epsilon_{mn}m_A + \Delta\epsilon_{mn}^2/2} \\ &= \frac{E'_v E_v}{E_v(1 - \cos\theta) + (m_A - \Delta\epsilon_{mn})}. \end{aligned}$$

Since before the integral is taken using delta function (234) the variables E'_v and $\cos\theta$ are still independent, all the dependence on $\cos\theta$ in the function $f(\cos\theta)$ from (236) is only through T_A , and for this reason the derivative in (234) is

$$\begin{aligned} \frac{df(\cos\theta_i)}{d\cos\theta} &= -\frac{dT_A}{d\cos\theta} \\ &= -\frac{d\sqrt{m_A^2 + E_v^2 + (E'_v)^2 - 2E'_v E_v \cos\theta}}{d\cos\theta} \\ &= \frac{E'_v E_v}{\sqrt{m_A^2 + E_v^2 + (E'_v)^2 - 2E'_v E_v \cos\theta}} = \frac{E'_v E_v}{T_A + m_A}. \end{aligned}$$

In view of this derivative, delta function (234) takes the form

$$\delta(f(\cos\theta)) = \delta(\cos\theta - \cos\theta_i) \frac{T_A(\cos\theta_i) + m_A}{E'_v E_v}.$$

Now cross section (224) can be integrated over $\cos\theta$ with this delta function

$$\begin{aligned} \frac{d\sigma_{mn}}{dE'_v} &= \int d\cos\theta \frac{(-1)E'_v |i\mathcal{M}_{mn}|^2}{2^5 \pi E_v (m_A + \epsilon_n)} \\ &\times \frac{\delta(E_v - E'_v - T_A(\cos\theta) - \Delta\epsilon_{mn})}{m_A + T_A + \epsilon_m} \\ &= \int d\cos\theta \frac{(-1)E'_v |i\mathcal{M}_{mn}|^2}{2^5 \pi E_v (m_A + \epsilon_n)} \\ &\times \frac{\delta(\cos\theta - \cos\theta_i)}{m_A + T_A(\cos\theta_i) + \epsilon_m} \frac{T_A(\cos\theta_i) + m_A}{E'_v E_v} \\ &= \frac{(-1)|i\mathcal{M}_{mn}|^2}{2^5 \pi E_v^2 (m_A + \epsilon_n) (m_A + T_A + \epsilon_m)} \frac{T_A + m_A}{E'_v}. \end{aligned}$$

A transition should be made from the unmeasurable variable E'_v to the measurable variable T_A , considering that $\cos\theta_i$ entering into the definition of T_A also depends on E'_v according to (235). Thus,

$$\begin{aligned} \frac{dT_A(\cos\theta_i)}{dE'_v} &= \frac{d\sqrt{m_A^2 + E_v^2 + (E'_v)^2 - 2E'_v E_v \cos\theta}}{dE'_v} \\ &= \frac{\frac{d(E'_v)^2}{dE'_v} - 2d(E'_v E_v \cos\theta)}{2\sqrt{m_A^2 + E_v^2 + (E'_v)^2 - 2E'_v E_v \cos\theta}} \end{aligned}$$

$$\begin{aligned} &= \frac{E'_v - E_v \cos\theta - E'_v E_v \frac{d\cos\theta}{dE'_v}}{\sqrt{m_A^2 + E_v^2 + (E'_v)^2 - 2E'_v E_v \cos\theta}} \\ &= \frac{E'_v - E_v \cos\theta - E'_v E_v \frac{d\cos\theta}{dE'_v}}{T_A + m_A}. \end{aligned}$$

Considering the derivative $\frac{dT_A(\cos\theta_i)}{dE'_v} = \frac{E_v(1 - \cos\theta) + (m_A - \Delta\epsilon_{mn})}{E'_v E_v}$ and recalling

that from (236) it follows that $E'_v - E_v + \Delta\epsilon_{mn} = -T_A$, we have

$$\begin{aligned} \frac{dT_A(\cos\theta_i)}{dE'_v} &= \frac{E_v(1 - \cos\theta) + (m_A - \Delta\epsilon_{mn})}{E'_v E_v} T_A + m_A \\ &= \frac{E'_v - E_v \cos\theta - E_v + E_v \cos\theta - m_A + \Delta\epsilon_{mn}}{T_A + m_A} \\ &= \frac{E'_v - E_v + \Delta\epsilon_{mn} - m_A}{T_A + m_A} = -1. \end{aligned}$$

This “-1” can also be obtained by differentiating (236) with respect to $\cos\theta_i$, that is, from $-dE'_v - dT_A = 0$. As a result, in view of this derivative, a known formula is obtained for the νA scattering cross section

$$\begin{aligned} \frac{d\sigma_{mn}}{dT_A} &= \frac{d\sigma_{mn}}{dE'_v} \frac{dE'_v}{dT_A(\cos\theta_i)} = \frac{(-1)|i\mathcal{M}_{mn}|^2}{2^5 \pi E_v^2 (m_A + \epsilon_n)} \\ &\times \frac{T_A + m_A}{m_A + T_A + \epsilon_m} (-1) = \frac{|i\mathcal{M}_{mn}|^2}{2^5 \pi E_v^2 m_A} C_{2,mn}, \end{aligned} \quad (237)$$

where $C_{2,mn} = \left[1 + \frac{\epsilon_n}{m_A}\right]^{-1} \left[1 + \frac{T_A}{m_A}\right] \left[1 + \frac{T_A + \epsilon_m}{m_A}\right]^{-1}$, which coincides with expression (233) derived by integrating the delta function over E'_v in cross section formula (224).

Thus, (i) integration of the conservation law delta function $\delta(E_v - E'_v - T_A(E'_v) - \Delta\epsilon_{mn})$ in the differential cross section (224) for νA scattering yields the same result irrespective of over which main variable, E'_v or $\cos\theta$, it is performed, and (ii) the corresponding expression for the coefficient $C_{2,mn}$ from [19] appears to be inaccurate, though nothing actually depends on the particular value of this coefficient.

10.4. Calculation of Scalar Products of Currents

The differential cross section for lepton–nucleus scattering (263) requires calculation of scalar products

of two currents, leptonic (in this case, neutrino) $\bar{u}(k', s') O^\mu u(k, s)$ and nucleon $\bar{u}(p', r') O'_\mu u(p, r)$, where

O_μ, O'_μ is the combination of the Dirac matrices. Unfortunately, the popular standard procedure of calculating matrix element by “taking” traces of γ matrices through summation over spin indices of the initial, s, r, s', r' , fermions is hardly suitable in this case. The reason is that it is impossible to use the known formula for Dirac spinors

$$u(\mathbf{p}, r) \bar{u}(\mathbf{p}, r) = \frac{1}{2} (\not{p} + m) (1 + \gamma_5 \not{s}_r),$$

where s_r denotes the four-vector of the fermion spin, since, at least in the case of coherent scattering, the spin projects of the nucleon in the initial and final state should be sharply controlled. In other words, scalar products are necessary for arbitrary combinations of final and initial spin projections of fermions.

Therefore, all scalar products are “directly” calculated (see, for example, [41]) by multiplying two above-mentioned current four-vectors.

To simplify intermediate computation, scalar products of the leptonic and nucleon currents will be obtained in the center-of-mass system of the neutral lepton and the nucleon, where the energies of the fermions entering and leaving the elastic scattering $v(k) + N(p) \rightarrow v(k') + N(p')$ are equal. For the (anti)neutrino and the nucleon, the respective energies are

$$E_v = \frac{s - m^2}{2\sqrt{s}}, \quad E_N = E_p = \frac{s + m^2}{2\sqrt{s}},$$

where $s = (p + k)^2$, p and m are the nucleon four-momentum and mass, and k is the (anti)neutrino four-momentum. Further on in this section all quantities depending on kinematic variables are given in the center-of-mass system of the (anti)neutrino and the nucleon. Kinematics of elastic scattering of the (anti)neutrino from the nucleon in their center-of-mass system can be constructed on the basis of the unit three-vector directed along the axis of the colliding particles, which can be chosen (without loss of generality) along the coordinate z axis (Fig. 21). Then directions of motion of all particles participating in the scattering will be given by unit vectors

$$\begin{aligned} \mathbf{n}_k &= (0, 0, 1), \quad \mathbf{n}_p = -\mathbf{n}_k, \\ \mathbf{n}_{k'} &= (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta), \quad \mathbf{n}_{p'} = -\mathbf{n}_{k'}. \end{aligned} \quad (238)$$

Here $\mathbf{n}_k \equiv \mathbf{n}_{k_i}$ and $\mathbf{n}_{k'} \equiv \mathbf{n}_{k_f}$ correspond to the initial and final (anti)neutrino, and $\mathbf{n}_p \equiv \mathbf{n}_{p_i}$ and $\mathbf{n}_{p'} \equiv \mathbf{n}_{p_f}$ correspond to the initial and final nucleon (proton or neutron). In the explicit form, they are

$$\begin{aligned} \mathbf{n}_{p_i} &= -\mathbf{n}_{k_i} = (0, 0, -1) \\ \mathbf{n}_{p_f} &= -\mathbf{n}_{k_f} = (-\cos \phi \sin \theta, -\sin \phi \sin \theta, -\cos \theta). \end{aligned}$$

It is convenient to calculate scalar products in different bases of two-component spinors χ_r . For example, summation over spin indices r, r' in the incoherent term of differential cross section (119) is easier to perform using the helical basis, where r, r' are eigenvalues of the helicity operator. The coherent term of cross section (263) requires consideration of the nucleon current retaining the nucleon spin projection on the given direction. The most convenient for solving this problem is the basis of bispinors χ_r , which are eigenstates of the operator $\sigma_3 = (\mathbf{n}_k \cdot \boldsymbol{\sigma})$. It is obvious that physical observables should not depend on the choice of the basis.

10.4.1. Helicity basis: Eigenvectors and components of currents. Let us begin with defining basis vectors of the (anti)neutrino and nucleon helicity operator $\hat{H}_{p_j/k_j} \equiv \mathbf{n}_{p_j/k_j} \cdot \boldsymbol{\sigma}$ where standard Pauli matrices have the form

$$\begin{aligned} \boldsymbol{\sigma} &= (\sigma_1, \sigma_2, \sigma_3); \quad \sigma_1 = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \sigma_2 &= \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

To this end, we should solve the helicity operator eigenvalue problem

$$\hat{H}_{p_j/k_j} \chi_r(\mathbf{n}_{p_j/k_j}) = r \chi_r(\mathbf{n}_{p_j/k_j}). \quad (239)$$

As is known, the two-component spinor $\chi_r(\mathbf{n})$ is an eigenvector of this kind of operator with the eigenvalue $r = \pm 1$ known as helicity or doubled spin projection of the fermion on its three-momentum.

Indeed, according to the direction of motion of the initial lepton along the third (or z) axis (238), the helicity operator has a simple form²⁷

$$\hat{H}_{k_i} = \mathbf{n}_{k_i} \cdot \boldsymbol{\sigma} = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The eigenvalue problem for the initial lepton helicity operator

$$\hat{H}_{k_i} \chi_{\mp}(\mathbf{n}_{k_i}) \equiv \sigma_3 \chi_{\mp}(\mathbf{n}_{k_i}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \chi_{\mp}(\mathbf{n}_{k_i}) = \mp \chi_{\mp}(\mathbf{n}_{k_i})$$

for the eigenvalues $r = -1$ and $r = +1$, when the spin is antiparallel ($\mathbf{s} \downarrow \uparrow \mathbf{k}_i$) and parallel ($\mathbf{s} \uparrow \uparrow \mathbf{k}_i$) to the momentum, has the solution in the form²⁸

²⁷ Since $\mathbf{n}_{k_i} \cdot \boldsymbol{\sigma} = (0, 0, 1) \cdot \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

²⁸ Because $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = +\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Fig. 21. Arrangement of unit vectors in the neutrino–nucleon center-of-mass system (left). Spins of the particles are shown on the right. For massless neutrinos (antineutrinos), initial and final projections of their spins ($\mathbf{s}_v$ and $\mathbf{s}'_v$) are always antiparallel (parallel) to their momenta ($\mathbf{k}_i$ and $\mathbf{k}_f$). For massive nucleons, three-vectors of their spins ($\mathbf{r}$ and $\mathbf{r}'$) can, generally speaking, be arbitrarily directed with respect to their three-momenta ($\mathbf{p}_i$ and $\mathbf{p}_f$).

$$\begin{aligned}\chi_{-}(\mathbf{k}_i) &\equiv \chi_{-}(\mathbf{n}_{k_i}) = \chi_{-1}(\mathbf{n}_{k_i}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ \chi_{+}(\mathbf{k}_i) &\equiv \chi_{+}(\mathbf{n}_{k_i}) = \chi_{+1}(\mathbf{n}_{k_i}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.\end{aligned}\quad (240)$$

Next, according to (238), the direction of motion of the initial nucleon in the center-of-mass system is opposite to the direction of motion of the initial lepton, i.e., along the $-z$ axis, and therefore the helicity operator for the initial nucleon is

$$\hat{H}_{p_i} = \mathbf{n}_{p_i} \cdot \boldsymbol{\sigma} = -\sigma_3 = -\hat{H}_{k_i}.$$

The eigenvalue problem for the initial nucleon helicity operator

$$\begin{aligned}\hat{H}_{p_i} \chi_{\mp}(\mathbf{n}_{p_i}) &\equiv [-\sigma_3] \chi_{\mp}(\mathbf{n}_{p_i}) \\ &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \chi_{\mp}(\mathbf{n}_{p_i}) = \mp \chi_{\mp}(\mathbf{n}_{p_i})\end{aligned}\quad (241)$$

for the eigenvalues $r = -1$ and $r = +1$ corresponding to the cases where the spin of the nucleon is antiparallel ($\mathbf{r} \downarrow \uparrow \mathbf{p}_i$) and parallel ($\mathbf{r} \uparrow \uparrow \mathbf{p}_i$) to its momentum has the solution in the form

$$\chi_{-}(\mathbf{p}_i) \equiv \chi_{-}(\mathbf{n}_{p_i}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (242)$$

$$\chi_{+}(\mathbf{p}_i) \equiv \chi_{+}(\mathbf{n}_{p_i}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ or } \chi_{+}(\mathbf{n}_{p_i}) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

In the latter case, any of the equivalent alternatives, with $+1$ or -1 , can be chosen.

Now we proceed to finding eigenvectors of helicity operators for final fermions. According to (238), the helicity operator for the final lepton moving in the direction $\mathbf{n}_{k_f}$ has the form

$$\hat{H}_{k_f} = \mathbf{n}_{k_f} \cdot \boldsymbol{\sigma} = \begin{pmatrix} \cos \theta & e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & -\cos \theta \end{pmatrix},$$

which can be shown by the direct calculation²⁹

$$\begin{aligned}\mathbf{n}_{k_f} \cdot \boldsymbol{\sigma} &= (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta) \cdot \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] \\ &= \begin{pmatrix} 0 & \cos \phi \sin \theta \\ \cos \phi \sin \theta & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \sin \phi \sin \theta \\ i \sin \phi \sin \theta & 0 \end{pmatrix} + \begin{pmatrix} \cos \theta & 0 \\ 0 & -\cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & -\cos \theta \end{pmatrix}.\end{aligned}$$

Then the eigenvalue problem for the final lepton helicity operator

$$\begin{aligned}\hat{H}_{k_f} \chi_{\mp}(\mathbf{n}_{k_f}) &= \begin{pmatrix} \cos \theta & e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & -\cos \theta \end{pmatrix} \\ &\times \chi_{\mp}(\mathbf{n}_{k_f}) = \mp \chi_{\mp}(\mathbf{n}_{k_f})\end{aligned}$$

$$\begin{aligned}\chi_{-}(\mathbf{k}_f) &\equiv \chi_{-}(\mathbf{n}_{k_f}) = \begin{pmatrix} -e^{-i\phi} \tan \frac{\theta}{2} \\ 1 \end{pmatrix}, \\ \chi_{+}(\mathbf{k}_f) &\equiv \chi_{+}(\mathbf{n}_{k_f}) = \begin{pmatrix} e^{-i\phi} \cot \frac{\theta}{2} \\ 1 \end{pmatrix}.\end{aligned}\quad (243)$$

for the eigenvalues $r = -1$ and $r = +1$, when the spin is antiparallel ($\mathbf{s}' \downarrow \uparrow \mathbf{k}_f$) and parallel ($\mathbf{s}' \uparrow \uparrow \mathbf{k}_f$) to the lepton momentum, has the solution in the form

Indeed, direct substitution of relations from footnote 44) gives

²⁹ From here on the following trigonometric relations are used: $e^{i\phi} = \cos \phi + i \sin \phi \rightarrow -i \cos \phi + \sin \phi = -ie^{i\phi}$, $i \cos \phi + \sin \phi = ie^{-i\phi}$.
 $\cos \theta = \sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$, $\tan \frac{\theta}{2} \sin \theta + \cos \theta = 1$, $\cot \frac{\theta}{2} \sin \theta - \cos \theta = 1$.

$$\begin{aligned}
\hat{H}_{k_f} \chi_{-}(\mathbf{n}_{k_f}) &= \begin{pmatrix} \cos \theta & e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & -\cos \theta \end{pmatrix} \begin{pmatrix} -e^{-i\phi} \tan \frac{\theta}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} -e^{-i\phi} \tan \frac{\theta}{2} \cos \theta + e^{-i\phi} \sin \theta \\ -e^{-i\phi} \tan \frac{\theta}{2} e^{i\phi} \sin \theta - \cos \theta \end{pmatrix} \\
&= \begin{pmatrix} -e^{-i\phi} [\tan \frac{\theta}{2} \cos \theta - \sin \theta] \\ -\tan \frac{\theta}{2} \sin \theta - \cos \theta \end{pmatrix} = (-1) \begin{pmatrix} e^{-i\phi} \tan \frac{\theta}{2} [\cos \theta - \sin \theta \cot \frac{\theta}{2}] \\ \tan \frac{\theta}{2} \sin \theta + \cos \theta \end{pmatrix} = (-1) \begin{pmatrix} -e^{-i\phi} \tan \frac{\theta}{2} \\ 1 \end{pmatrix}; \\
\hat{H}_{k_f} \chi_{+}(\mathbf{n}_{k_f}) &= \begin{pmatrix} \cos \theta & e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & -\cos \theta \end{pmatrix} \begin{pmatrix} e^{-i\phi} \cot \frac{\theta}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} \cos \theta e^{-i\phi} \cot \frac{\theta}{2} + e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta e^{-i\phi} \cot \frac{\theta}{2} - \cos \theta \end{pmatrix} \\
&= \begin{pmatrix} e^{-i\phi} [\cos \theta \cot \frac{\theta}{2} + \sin \theta] \\ \sin \theta \cot \frac{\theta}{2} - \cos \theta \end{pmatrix} = \begin{pmatrix} e^{-i\phi} \cot \frac{\theta}{2} [\cos \theta + \sin \theta \tan \frac{\theta}{2}] \\ \sin \theta \cot \frac{\theta}{2} - \cos \theta \end{pmatrix} = (+1) \chi_{+}(\mathbf{n}_{k_f}).
\end{aligned}$$

Since the direction of the emission of the final nucleon is opposite to the direction of the emission of the final lepton, the matrix of the final nucleon helicity operator is $H_{p_f} = -H_{k_f}$. The solution of the corresponding eigenvalue problem for the final nucleon

$$\begin{aligned}
\chi_{+}^{\dagger}(\mathbf{k}_f) \chi_{+}(\mathbf{k}_f) &= \begin{pmatrix} e^{-i\phi} \cot \frac{\theta}{2} \\ 1 \end{pmatrix}^{\dagger} \begin{pmatrix} e^{-i\phi} \cot \frac{\theta}{2} \\ 1 \end{pmatrix} \\
&= \cot^2 \frac{\theta}{2} + 1 = \sin^{-2} \frac{\theta}{2}.
\end{aligned}$$

$$\begin{aligned}
\hat{H}_{p_f} \chi_{\mp}(\mathbf{n}_{p_f}) &\equiv \begin{pmatrix} -\cos \theta & -e^{-i\phi} \sin \theta \\ -e^{i\phi} \sin \theta & \cos \theta \end{pmatrix} \\
&\times \chi_{\mp}(\mathbf{n}_{p_f}) = \mp \chi_{\mp}(\mathbf{n}_{p_f})
\end{aligned} \quad (244)$$

for $r = -1$ (when $(\mathbf{r}' \downarrow \uparrow \mathbf{p}_f)$) and $r = +1$ (when $(\mathbf{r}' \uparrow \uparrow \mathbf{p}_f)$) has the form

$$\begin{aligned}
\chi_{-}(\mathbf{p}_f) \equiv \chi_{-}(\mathbf{n}_{p_f}) &= \begin{pmatrix} e^{-i\phi} \cot \frac{\theta}{2} \\ 1 \end{pmatrix}, \\
\chi_{+}(\mathbf{p}_f) \equiv \chi_{+}(\mathbf{n}_{p_f}) &= \begin{pmatrix} -e^{-i\phi} \tan \frac{\theta}{2} \\ 1 \end{pmatrix}.
\end{aligned} \quad (245)$$

Expressions (240), (242), (243), and (245) involve a complete set of eigenstates of helicity operators. Obviously, eigenstates of the initial fermion helicity operator (240) and (242) are already normalized to unity, i.e., $|\chi_{\pm}(\mathbf{p}/\mathbf{k}_f)|^2 = 1$, whereas eigenstates of the final fermion helicity operators in the form (243) and (245) do not have this property, i.e., $|\chi_{\pm}(\mathbf{p}/\mathbf{k}_f)|^2 \neq 1$. Indeed,

$$\begin{aligned}
\chi_{-}^{\dagger}(\mathbf{k}_f) \chi_{-}(\mathbf{k}_f) &= \begin{pmatrix} -e^{-i\phi} \tan \frac{\theta}{2} \\ 1 \end{pmatrix}^{\dagger} \begin{pmatrix} -e^{-i\phi} \tan \frac{\theta}{2} \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} -e^{+i\phi} \tan \frac{\theta}{2} & 1 \end{pmatrix} \begin{pmatrix} -e^{-i\phi} \tan \frac{\theta}{2} \\ 1 \end{pmatrix} = \cos^{-2} \frac{\theta}{2};
\end{aligned}$$

It is evident from the formulation of the eigenvalue problem that if $\chi_{+}(\mathbf{p}_f)$ is the eigenvector of the helicity operator for the eigenvalue $+1$, then $\tilde{\chi}_{+}(\mathbf{p}_f) = a \chi_{+}(\mathbf{p}_f)$ is also the eigenvector for the eigenvalue $+1$, where a is an arbitrary factor. Therefore, taking into account that, according to (243) and (245), $\chi_{-}^{\dagger}(\mathbf{k}_f) \chi_{-}(\mathbf{k}_f) = \chi_{+}^{\dagger}(\mathbf{p}_f) \chi_{+}(\mathbf{p}_f)$ and $\chi_{+}^{\dagger}(\mathbf{k}_f) \chi_{+}(\mathbf{k}_f) = \chi_{-}^{\dagger}(\mathbf{p}_f) \chi_{-}(\mathbf{p}_f)$, redefining two-spinors, for example, in the form

$$\begin{aligned}
\tilde{\chi}_{-}(\mathbf{k}_f) &= \cos \frac{\theta}{2} \chi_{-}(\mathbf{k}_f), \\
\tilde{\chi}_{+}(\mathbf{k}_f) &= e^{i\phi} \sin \frac{\theta}{2} \chi_{+}(\mathbf{k}_f), \\
\tilde{\chi}_{-}(\mathbf{p}_f) &= \sin \frac{\theta}{2} \chi_{-}(\mathbf{p}_f), \\
\tilde{\chi}_{+}(\mathbf{p}_f) &= -e^{i\phi} \cos \frac{\theta}{2} \chi_{+}(\mathbf{p}_f)
\end{aligned}$$

and omitting tildes, we will obtain them as satisfying the condition $|\chi_{\pm}(\mathbf{p}/\mathbf{k}_{i/f})|^2 = 1$.

Thus, normalized-to-unity eigenvectors (two-component spinors) of the helicity operators for the initial and final (anti)neutrinos and nucleons in their center-of-mass system have the form

$$\begin{aligned}
\chi_{-}(\mathbf{k}_i) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\
\chi_{-}(\mathbf{k}_f) &= \cos \frac{\theta}{2} \begin{pmatrix} -e^{-i\phi} \tan \frac{\theta}{2} \\ 1 \end{pmatrix} \begin{pmatrix} -e^{-i\phi} \sin \frac{\theta}{2} \\ \cos \frac{\theta}{2} \end{pmatrix}, \\
\chi_{+}(\mathbf{k}_i) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\
\chi_{+}(\mathbf{k}_f) &= e^{i\phi} \sin \frac{\theta}{2} \begin{pmatrix} e^{-i\phi} \cot \frac{\theta}{2} \\ 1 \end{pmatrix} \begin{pmatrix} \cos \frac{\theta}{2} \\ e^{i\phi} \sin \frac{\theta}{2} \end{pmatrix}, \\
\chi_{-}(\mathbf{p}_i) &= \begin{pmatrix} 1 \\ 0(81) \end{pmatrix}, \\
\chi_{-}(\mathbf{p}_f) &= \sin \frac{\theta}{2} \begin{pmatrix} e^{-i\phi} \cot \frac{\theta}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} e^{-i\phi} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \end{pmatrix}, \\
\chi_{+}(\mathbf{p}_i) &= \begin{pmatrix} 0 \\ 1(82) \end{pmatrix}, \\
\chi_{+}(\mathbf{p}_f) &= -e^{i\phi} \cos \frac{\theta}{2} \begin{pmatrix} -e^{-i\phi} \tan \frac{\theta}{2} \\ 1 \end{pmatrix} \begin{pmatrix} \sin \frac{\theta}{2} \\ -e^{i\phi} \cos \frac{\theta}{2} \end{pmatrix}.
\end{aligned} \tag{246}$$

In [19], the bispinor of the initial nucleon $\chi_{+}(\mathbf{n}_p)$ is defined with a different sign

$$\begin{aligned}
\chi_{+}(\mathbf{n}_k) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_{+}(\mathbf{n}_{k'}) = \begin{pmatrix} \cos \frac{\theta}{2}(84) \\ e^{i\phi} \sin \frac{\theta}{2} \end{pmatrix}, \\
\chi_{-}(\mathbf{n}_k) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \chi_{-}(\mathbf{n}_{k'}) = \begin{pmatrix} -e^{-i\phi} \sin \frac{\theta}{2} \\ \cos \frac{\theta}{2} \end{pmatrix}, \\
\chi_{+}(\mathbf{n}_p) &= \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad \chi_{+}(\mathbf{n}_{p'}) = \begin{pmatrix} \sin \frac{\theta}{2} \\ -e^{i\phi} \cos \frac{\theta}{2} \end{pmatrix}, \\
\chi_{-}(\mathbf{n}_p) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_{-}(\mathbf{n}_{p'}) = \begin{pmatrix} e^{-i\phi} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \end{pmatrix}.
\end{aligned} \tag{247}$$

Obviously, the ultimate result does not depend on the difference between (246) and (247).

Helicity basis: Current components

In the basis of the Dirac γ matrices $\gamma^\mu = (\gamma^0, \gamma)$, $\gamma_\mu = (\gamma_0, -\gamma) = g_{\mu\nu} \gamma^\nu$, where

$$\begin{aligned}
\gamma_0 = \gamma^0 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_5 = \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\gamma &= \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix},
\end{aligned}$$

the four-spinor with three-momentum $\mathbf{p}$ and spin index $r = \pm 1$, which is a solution to the Dirac equation, for the massive fermion (nucleon) has, according to (169), the form

$$u(\mathbf{p}, r) = \begin{pmatrix} \lambda_{+} \\ \lambda_{-} \alpha_p \end{pmatrix} \chi_r(\mathbf{p}). \tag{248}$$

The index r numbers two linearly independent bispinors $\chi_r(\mathbf{p})$. Accordingly, the four-spinor with three-momentum $\mathbf{k}$ and spin index $s = \pm 1$ for the massive antifermion has, according to (169), the form

$$v(\mathbf{k}, s) = \begin{pmatrix} \lambda_{-} \alpha_k \\ \lambda_{+} \end{pmatrix} \chi_s(\mathbf{k}), \tag{249}$$

where $\lambda_{\pm} = \sqrt{E_N \pm m}$ and $\alpha_p = \mathbf{n}_p \cdot \boldsymbol{\sigma} = \mathbf{n}_{p_i} \cdot \boldsymbol{\sigma}$; $\alpha_k = \mathbf{n}_k \cdot \boldsymbol{\sigma} = \mathbf{n}_{k_i} \cdot \boldsymbol{\sigma}$. Further, we will need combinations of bispinors and Pauli matrices following from (248) and (249)

$$u^\dagger(\mathbf{p}', r') = \chi_{r'}^\dagger(\mathbf{p}') (\lambda_{+}, \lambda_{-} \alpha_{p'}^\dagger),$$

$$v^\dagger(\mathbf{k}', s') = \chi_{s'}^\dagger(\mathbf{k}') (\lambda_{-} \alpha_{k'}^\dagger, \lambda_{+});$$

$$\begin{aligned}
\bar{u}(\mathbf{p}', r') &= u^\dagger(\mathbf{p}', r') \gamma_0 \\
&= \chi_{r'}^\dagger(\mathbf{p}') (\lambda_{+}, \lambda_{-} \alpha_{p'}^\dagger) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= \chi_{r'}^\dagger(\mathbf{p}') (\lambda_{+}, -\lambda_{-} \alpha_{p'}^\dagger), \\
\bar{u}(\mathbf{p}', r') \gamma_0 &= u^\dagger(\mathbf{p}', r') \gamma_0^2 = u^\dagger(\mathbf{p}', r') \\
&= \chi_{r'}^\dagger(\mathbf{p}') (\lambda_{+}, \lambda_{-} \alpha_{p'}^\dagger),
\end{aligned} \tag{250}$$

$$\begin{aligned}
\bar{u}(\mathbf{p}', r') \gamma &= \chi_{r'}^\dagger(\mathbf{p}') (\lambda_{+}, -\lambda_{-} \alpha_{p'}^\dagger) \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix} \\
&= \chi_{r'}^\dagger(\mathbf{p}') (\lambda_{-} \alpha_{p'}^\dagger \sigma, \lambda_{+} \sigma),
\end{aligned}$$

$$\begin{aligned}
\bar{u}(\mathbf{p}', r') \gamma_0 \gamma_5 &= \chi_{r'}^\dagger(\mathbf{p}') (\lambda_{+}, \lambda_{-} \alpha_{p'}^\dagger) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \chi_{r'}^\dagger(\mathbf{p}') (\lambda_{-} \alpha_{p'}^\dagger, \lambda_{+}),
\end{aligned}$$

$$\begin{aligned}
\bar{u}(\mathbf{p}', r') \gamma \gamma_5 &= \chi_{r'}^\dagger(\mathbf{p}') (\lambda_{-} \alpha_{p'}^\dagger \sigma, \lambda_{+} \sigma) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \chi_{r'}^\dagger(\mathbf{p}') (\lambda_{+} \sigma, \lambda_{-} \alpha_{p'}^\dagger \sigma).
\end{aligned}$$

Here $\alpha_{k'/p'} = \mathbf{n}_{k'/p'} \cdot \boldsymbol{\sigma} = \mathbf{n}_{k_f/p_f} \cdot \boldsymbol{\sigma}$ and $\alpha_{k'/p'}^\dagger = \alpha_{k'/p'}$, since $\sigma^\dagger = \sigma$.

With expressions (250), all components corresponding to $\mu = 0, 1, 2, 3$ of the vector and axial-vector nucleon currents, which are given by the formulas

$$\begin{aligned} V_{r'r}^\mu(\mathbf{p}', \mathbf{p}) &\equiv \bar{u}(\mathbf{p}', r') \gamma^\mu u(\mathbf{p}, r), \\ A_{r'r}^\mu(\mathbf{p}', \mathbf{p}) &\equiv \bar{u}(\mathbf{p}', r') \gamma^\mu \gamma^5 u(\mathbf{p}, r), \end{aligned} \quad (251)$$

for all helicities and momenta of the initial ($r = \pm, \mathbf{p}$) and final ($r' = \pm, \mathbf{p}'$) nucleon can be expressed in terms of the eigenstates of the nucleon helicity operators

$$\begin{aligned} V_{r'r}^0(\mathbf{p}', \mathbf{p}) &= \chi_{r'}^\dagger(\mathbf{n}_p) [\lambda_+^2 + \lambda_-^2 \alpha_p \alpha_p] \chi_r(\mathbf{n}_p) \\ &= \chi_{r'}^\dagger(\mathbf{p}') \hat{V}^0(\mathbf{p}', \mathbf{p}) \chi_r(\mathbf{p}), \\ \mathbf{V}_{r'r}(\mathbf{p}', \mathbf{p}) &= \chi_{r'}^\dagger(\mathbf{n}_p) \lambda_+ \lambda_- [\sigma \alpha_p + \alpha_p \sigma] \chi_r(\mathbf{n}_p) \\ &= \chi_{r'}^\dagger(\mathbf{p}') \hat{\mathbf{V}}(\mathbf{p}', \mathbf{p}) \chi_r(\mathbf{p}), \\ A_{r'r}^0(\mathbf{p}', \mathbf{p}) &= \chi_{r'}^\dagger(\mathbf{n}_p) \lambda_+ \lambda_- [\alpha_p + \alpha_p] \chi_r(\mathbf{n}_p) \\ &= \chi_{r'}^\dagger(\mathbf{p}') \hat{A}^0(\mathbf{p}', \mathbf{p}) \chi_r(\mathbf{p}), \\ \mathbf{A}_{r'r}(\mathbf{p}', \mathbf{p}) &= \chi_{r'}^\dagger(\mathbf{n}_p) [\lambda_+^2 \sigma + \lambda_-^2 \alpha_p \sigma \alpha_p] \chi_r(\mathbf{n}_p) \\ &= \chi_{r'}^\dagger(\mathbf{p}') \hat{\mathbf{A}}(\mathbf{p}', \mathbf{p}) \chi_r(\mathbf{p}). \end{aligned} \quad (252)$$

Here “current” matrices are introduced,

$$\begin{aligned} \hat{V}^0(\mathbf{p}', \mathbf{p}) &\equiv \lambda_+^2 + \lambda_-^2 \alpha_p^\dagger \alpha_p, \\ \hat{\mathbf{V}}(\mathbf{p}', \mathbf{p}) &\equiv \lambda_- \lambda_+ [\alpha_p^\dagger \sigma + \sigma \alpha_p], \\ \hat{A}^0(\mathbf{p}', \mathbf{p}) &\equiv \lambda_+ \lambda_- [\alpha_p^\dagger + \alpha_p], \\ \hat{\mathbf{A}}(\mathbf{p}', \mathbf{p}) &\equiv \lambda_+^2 \sigma + \lambda_-^2 \alpha_p^\dagger \sigma \alpha_p. \end{aligned} \quad (253)$$

Substituting bispinors (250) into currents (251), we obtain formulas (252)

$$\begin{aligned} V_{r'r}^0(\mathbf{p}', \mathbf{p}) &= \bar{u}(\mathbf{p}', r') \gamma_0 u(\mathbf{p}, r) \\ &= \chi_{r'}^\dagger(\mathbf{p}') (\lambda_+, \lambda_- \alpha_p^\dagger) \begin{pmatrix} \lambda_+ \\ \lambda_- \alpha_p \end{pmatrix} \chi_r(\mathbf{p}) \\ &= \chi_{r'}^\dagger(\mathbf{p}') (\lambda_+^2 + \lambda_-^2 \alpha_p^\dagger \alpha_p) \chi_r(\mathbf{p}); \\ \mathbf{V}_{r'r}(\mathbf{p}', \mathbf{p}) &= \bar{u}(\mathbf{p}', r') \boldsymbol{\gamma} u(\mathbf{p}, r) \\ &= \chi_{r'}^\dagger(\mathbf{p}') (\lambda_- \alpha_p^\dagger \sigma, \lambda_+ \sigma) \begin{pmatrix} \lambda_+ \\ \lambda_- \alpha_p \end{pmatrix} \chi_r(\mathbf{p}) \\ &= \lambda_- \lambda_+ \chi_{r'}^\dagger(\mathbf{p}') (\alpha_p^\dagger \sigma + \sigma \alpha_p) \chi_r(\mathbf{p}); \\ A_{r'r}^0(\mathbf{p}', \mathbf{p}) &= \bar{u}(\mathbf{p}', r') \gamma_5 u(\mathbf{p}, r) \\ &= \chi_{r'}^\dagger(\mathbf{p}') (\lambda_- \alpha_p^\dagger, \lambda_+) \begin{pmatrix} \lambda_+ \\ \lambda_- \alpha_p \end{pmatrix} \chi_r(\mathbf{p}) \\ &= \lambda_+ \lambda_- \chi_{r'}^\dagger(\mathbf{p}') (\alpha_p^\dagger + \alpha_p) \chi_r(\mathbf{p}); \\ \mathbf{A}_{r'r}(\mathbf{p}', \mathbf{p}) &= \bar{u}(\mathbf{p}', r') \boldsymbol{\gamma} \gamma_5 u(\mathbf{p}, r) \\ &= \chi_{r'}^\dagger(\mathbf{p}') (\lambda_+ \sigma, \lambda_- \alpha_p^\dagger \sigma) \begin{pmatrix} \lambda_+ \\ \lambda_- \alpha_p \end{pmatrix} \chi_r(\mathbf{p}) \\ &= \chi_{r'}^\dagger(\mathbf{p}') [\lambda_+^2 \sigma + \lambda_-^2 \alpha_p^\dagger \sigma \alpha_p] \chi_r(\mathbf{p}). \end{aligned}$$

If in Eqs. (248), (251), and (252) one substitutes $\mathbf{k}, \mathbf{k}'$ for $\mathbf{p}, \mathbf{p}'$ by considering $\alpha_k = \mathbf{n}_k \cdot \boldsymbol{\sigma} = \mathbf{n}_{k'} \cdot \boldsymbol{\sigma}$ and

$\alpha_{k'} = \mathbf{n}_{k'} \cdot \boldsymbol{\sigma} = \mathbf{n}_{k'} \cdot \boldsymbol{\sigma}$, one can obtain an analog of Eqs. (252) for the neutrino

$$\begin{aligned} V_{r'r}^0(\mathbf{k}', \mathbf{k}) &= \bar{u}(\mathbf{k}', r') \gamma_0 u(\mathbf{k}, r) \\ &= \chi_{r'}^\dagger(\mathbf{k}') \hat{V}^0(\mathbf{k}', \mathbf{k}) \chi_r(\mathbf{k}), \\ \mathbf{V}_{r'r}(\mathbf{k}', \mathbf{k}) &= \bar{u}(\mathbf{k}', r') \boldsymbol{\gamma} u(\mathbf{k}, r) \\ &= \chi_{r'}^\dagger(\mathbf{k}') \hat{\mathbf{V}}(\mathbf{k}', \mathbf{k}) \chi_r(\mathbf{k}), \\ A_{r'r}^0(\mathbf{k}', \mathbf{k}) &= \bar{u}(\mathbf{k}', r') \gamma_5 u(\mathbf{k}, r) \\ &= \chi_{r'}^\dagger(\mathbf{k}') \hat{A}^0(\mathbf{k}', \mathbf{k}) \chi_r(\mathbf{k}), \\ \mathbf{A}_{r'r}(\mathbf{k}', \mathbf{k}) &= \bar{u}(\mathbf{k}', r') \boldsymbol{\gamma} \gamma_5 u(\mathbf{k}, r) \\ &= \chi_{r'}^\dagger(\mathbf{k}') \hat{\mathbf{A}}(\mathbf{k}', \mathbf{k}) \chi_r(\mathbf{k}), \end{aligned} \quad (254)$$

where the neutrino current matrices have the form

$$\begin{aligned} \hat{V}^0(\mathbf{k}', \mathbf{k}) &\equiv \lambda_+^2 + \lambda_-^2 \alpha_k^\dagger \alpha_k, \quad \hat{\mathbf{V}}(\mathbf{k}', \mathbf{k}) \equiv \lambda_- \lambda_+ [\alpha_k^\dagger \boldsymbol{\sigma} + \boldsymbol{\sigma} \alpha_k], \\ \hat{A}^0(\mathbf{k}', \mathbf{k}) &\equiv \lambda_+ \lambda_- [\alpha_k^\dagger + \alpha_k], \quad \hat{\mathbf{A}}(\mathbf{k}', \mathbf{k}) \equiv \lambda_+^2 \boldsymbol{\sigma} + \lambda_-^2 \alpha_k^\dagger \boldsymbol{\sigma} \alpha_k. \end{aligned}$$

Nevertheless, it is important to directly obtain the components of the neutrino and antineutrino currents. For this purpose, particle and antiparticle four-spinors, Dirac equation solutions (248) and (249) for $m = 0$ and correspondingly $\lambda = \sqrt{E_v}$, are used. They have the form

$$\begin{aligned} u(\mathbf{k}, r) &= \lambda \begin{pmatrix} 1 \\ \alpha_k \end{pmatrix} \chi_r(\mathbf{k}), \quad u^\dagger(\mathbf{k}', r') = \lambda \chi_{r'}^\dagger(\mathbf{k}') (1, \alpha_k^\dagger), \\ \bar{u}(\mathbf{k}', r') &= \lambda \chi_{r'}^\dagger(\mathbf{k}') (1, -\alpha_k^\dagger), \end{aligned} \quad (255)$$

$$v(\mathbf{k}, r) = \lambda \begin{pmatrix} \alpha_k \\ 1 \end{pmatrix} \chi_r(\mathbf{k}),$$

$$v^\dagger(\mathbf{k}', r') = \lambda \chi_{r'}^\dagger(\mathbf{k}') (\alpha_k^\dagger, 1), \quad (256)$$

$$\bar{v}(\mathbf{k}', r') = \lambda \chi_{r'}^\dagger(\mathbf{k}') (\alpha_k^\dagger, -1).$$

Expressions for the components of the neutrino vector and axial-vector currents are obtained from (254) without using (255) with the replacement $\lambda_\pm \rightarrow \lambda$

$$\begin{aligned} V_{r'r}^0(\mathbf{k}', \mathbf{k}) &= \bar{u}(\mathbf{k}', r') \gamma_0 u(\mathbf{k}, r) \\ &= \lambda^2 \chi_{r'}^\dagger(\mathbf{k}') [1 + \alpha_r^\dagger \alpha_k] \chi_r(\mathbf{k}), \\ \mathbf{V}_{r'r}(\mathbf{k}', \mathbf{k}) &= \bar{u}(\mathbf{k}', r') \boldsymbol{\gamma} u(\mathbf{k}, r) \\ &= \lambda^2 \chi_{r'}^\dagger(\mathbf{k}') [\alpha_r^\dagger \boldsymbol{\sigma} + \boldsymbol{\sigma} \alpha_k] \chi_r(\mathbf{k}), \\ A_{r'r}^0(\mathbf{k}', \mathbf{k}) &= \bar{u}(\mathbf{k}', r') \gamma_5 u(\mathbf{k}, r) \\ &= \lambda^2 \chi_{r'}^\dagger(\mathbf{k}') [\alpha_r^\dagger + \alpha_k] \chi_r(\mathbf{k}), \\ \mathbf{A}_{r'r}(\mathbf{k}', \mathbf{k}) &= \bar{u}(\mathbf{k}', r') \boldsymbol{\gamma} \gamma_5 u(\mathbf{k}, r) \\ &= \lambda^2 \chi_{r'}^\dagger(\mathbf{k}') [\boldsymbol{\sigma} + \alpha_r^\dagger \boldsymbol{\sigma} \alpha_k] \chi_r(\mathbf{k}). \end{aligned} \quad (257)$$

By analogy with formulas (250) for particle four-spinors, the rest of the necessary combinations of bispinors and Pauli matrices for the antiparticle (anti-

neutrino) can be constructed on the basis of the four-spinor (256). They are

$$\begin{aligned}
 \bar{\nu}(\mathbf{k}', r') \gamma_0 &= \nu^\dagger(\mathbf{k}', r') \gamma_0^2 = \nu^\dagger(\mathbf{k}', r') \\
 &= \lambda \chi_{r'}^\dagger(\mathbf{k}') (\alpha_k^\dagger, 1), \\
 \bar{\nu}(\mathbf{k}', r') \gamma &= \lambda \chi_{r'}^\dagger(\mathbf{k}') (\alpha_k^\dagger, -1) \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix} \\
 &= \lambda \chi_{r'}^\dagger(\mathbf{k}') (\sigma, \alpha_k^\dagger \sigma), \\
 \bar{\nu}(\mathbf{k}', r') \gamma_0 \gamma_5 &= \lambda \chi_{r'}^\dagger(\mathbf{k}') (\alpha_k^\dagger, 1) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
 &= \lambda \chi_{r'}^\dagger(\mathbf{k}') (1, \alpha_k^\dagger), \\
 \bar{\nu}(\mathbf{k}', r') \gamma \gamma_5 &= \lambda \chi_{r'}^\dagger(\mathbf{k}') (\sigma, \alpha_k^\dagger \sigma) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
 &= \lambda \chi_{r'}^\dagger(\mathbf{k}') (\alpha_k^\dagger \sigma, \sigma).
 \end{aligned} \tag{258}$$

With (258), the components of the antineutrino currents similar to (257) are obtained in the form

$$\begin{aligned}
 \bar{\nu}_{r',r}^0(\mathbf{k}', \mathbf{k}) &= \bar{\nu}(\mathbf{k}', r') \gamma_0 \nu(\mathbf{k}, r) \\
 &= \lambda^2 \chi_{r'}^\dagger(\mathbf{k}') (\alpha_k^\dagger, 1) \begin{pmatrix} \alpha_k \\ 1 \end{pmatrix} \chi_r(\mathbf{k}) \\
 &= \lambda^2 \chi_{r'}^\dagger(\mathbf{k}') [\alpha_k^\dagger \alpha_k + 1] \chi_r(\mathbf{k}), \\
 \bar{\mathbf{V}}_{r',r}(\mathbf{k}', \mathbf{k}) &= \bar{\nu}(\mathbf{k}', r') \gamma \nu(\mathbf{k}, r) \\
 &= \lambda^2 \chi_{r'}^\dagger(\mathbf{k}') (\sigma, \alpha_k^\dagger \sigma) \begin{pmatrix} \alpha_k \\ 1 \end{pmatrix} \chi_r(\mathbf{k}) \\
 &= \lambda^2 \chi_{r'}^\dagger(\mathbf{k}') [\sigma \alpha_k + \alpha_k^\dagger \sigma] \chi_r(\mathbf{k}), \\
 \bar{A}_{r',r}^0(\mathbf{k}', \mathbf{k}) &= \bar{\nu}(\mathbf{k}', r') \gamma_0 \gamma_5 \nu(\mathbf{k}, r) \\
 &= \lambda^2 \chi_{r'}^\dagger(\mathbf{k}') (1, \alpha_k^\dagger) \begin{pmatrix} \alpha_k \\ 1 \end{pmatrix} \chi_r(\mathbf{k}) \\
 &= \lambda^2 \chi_{r'}^\dagger(\mathbf{k}') [\alpha_k + \alpha_k^\dagger] \chi_r(\mathbf{k}), \\
 \bar{\mathbf{A}}_{r',r}(\mathbf{k}', \mathbf{k}) &= \bar{\nu}(\mathbf{k}', r') \gamma \gamma_5 \nu(\mathbf{k}, r) \\
 &= \lambda^2 \chi_{r'}^\dagger(\mathbf{k}') (\alpha_k^\dagger \sigma, \sigma) \begin{pmatrix} \alpha_k \\ 1 \end{pmatrix} \chi_r(\mathbf{k}) \\
 &= \lambda^2 \chi_{r'}^\dagger(\mathbf{k}') [\alpha_k^\dagger \sigma \alpha_k + \sigma] \chi_r(\mathbf{k}).
 \end{aligned} \tag{259}$$

Comparing (259) and (257) one can see that these expressions in terms of the bispinors $\chi_{r'}^\dagger(\mathbf{k}')$ and $\chi_r(\mathbf{k})$ coincide, because the matrices under the sign of sum can be permuted. Therefore, the neutrino and antineutrino V - and A -currents are written in the unified form

$$\begin{aligned}
 V_{r',r}^0(\mathbf{k}', \mathbf{k}) &= \bar{V}_{r',r}^0(\mathbf{k}', \mathbf{k}) = \chi_{r'}^\dagger(\mathbf{k}') \hat{\nu}^0(\mathbf{k}', \mathbf{k}) \chi_r(\mathbf{k}), \\
 \mathbf{V}_{r',r}(\mathbf{k}', \mathbf{k}) &= \bar{\mathbf{V}}_{r',r}(\mathbf{k}', \mathbf{k}) = \chi_{r'}^\dagger(\mathbf{k}') \hat{\mathbf{V}}(\mathbf{k}', \mathbf{k}) \chi_r(\mathbf{k}), \\
 A_{r',r}^0(\mathbf{k}', \mathbf{k}) &= \bar{A}_{r',r}^0(\mathbf{k}', \mathbf{k}) = \chi_{r'}^\dagger(\mathbf{k}') \hat{A}^0(\mathbf{k}', \mathbf{k}) \chi_r(\mathbf{k}), \\
 \mathbf{A}_{r',r}(\mathbf{k}', \mathbf{k}) &= \bar{\mathbf{A}}_{r',r}(\mathbf{k}', \mathbf{k}) = \chi_{r'}^\dagger(\mathbf{k}') \hat{\mathbf{A}}(\mathbf{k}', \mathbf{k}) \chi_r(\mathbf{k}),
 \end{aligned} \tag{260}$$

where the current matrices common for the neutrino and the antineutrino are introduced,

$$\begin{aligned}
 \hat{\nu}^0(\mathbf{k}', \mathbf{k}) &\equiv \lambda^2 [1 + \alpha_k^\dagger \alpha_k], \\
 \hat{\mathbf{V}}(\mathbf{k}', \mathbf{k}) &\equiv \lambda^2 [\alpha_k^\dagger \sigma + \sigma \alpha_k], \\
 \hat{A}^0(\mathbf{k}', \mathbf{k}) &\equiv \lambda^2 [\alpha_k^\dagger + \alpha_k], \\
 \hat{\mathbf{A}}(\mathbf{k}', \mathbf{k}) &\equiv \lambda^2 [\sigma + \alpha_k^\dagger \sigma \alpha_k].
 \end{aligned} \tag{261}$$

The difference between neutrino and antineutrino currents is due to the fact that (anti)neutrino chirality coincides with helicity and is preserved, from which it follows that for $m_\nu = m_{\bar{\nu}} = 0$ neutrinos always have left-handed helicity, and the bispinors $\chi_-^\dagger(\mathbf{k}')$ and $\chi_-(\mathbf{k})$ should be used for them in these expressions. Accordingly, antineutrinos always have right-handed helicity, and the bispinors $\chi_+^\dagger(\mathbf{k}')$ and $\chi_+(\mathbf{k})$ should be used for them.

Helicity basis: Components of leptonic currents

Since the neutrino always has left-handed helicity (the spin of the neutrino is directed against its momentum), the complete description of the neutrino requires only the bispinors $\chi_-(\mathbf{k}_i)$ and $\chi_-(\mathbf{k}_f)$ from (246). Therefore, the following components of the weak vector and axial currents should be calculated for the neutrino:

$$\begin{aligned}
 V_-^0(\mathbf{k}', \mathbf{k}) &= \chi_-^\dagger(\mathbf{k}') \hat{\nu}^0(\mathbf{k}', \mathbf{k}) \chi_-(\mathbf{k}), \\
 \mathbf{V}_-(\mathbf{k}', \mathbf{k}) &= \chi_-^\dagger(\mathbf{k}') \hat{\mathbf{V}}(\mathbf{k}', \mathbf{k}) \chi_-(\mathbf{k}), \\
 A_-^0(\mathbf{k}', \mathbf{k}) &= \chi_-^\dagger(\mathbf{k}') \hat{A}^0(\mathbf{k}', \mathbf{k}) \chi_-(\mathbf{k}), \\
 \mathbf{A}_-(\mathbf{k}', \mathbf{k}) &= \chi_-^\dagger(\mathbf{k}') \hat{\mathbf{A}}(\mathbf{k}', \mathbf{k}) \chi_-(\mathbf{k}).
 \end{aligned}$$

For antineutrinos, which always have right-handed helicity (i.e., the spin of the antineutrino is directed along its momentum), only the bispinors $\chi_+(\mathbf{k}_i)$ and $\chi_+(\mathbf{k}_f)$ from (246) are applicable, and it is necessary to calculate other components (additional to neutrino ones) of the weak vector and axial currents³⁰

$$\begin{aligned}
 \bar{V}_{++}^0(\mathbf{k}', \mathbf{k}) &= \chi_+^\dagger(\mathbf{k}') \hat{\nu}^0(\mathbf{k}', \mathbf{k}) \chi_+(\mathbf{k}), \\
 \bar{\mathbf{V}}_{++}(\mathbf{k}', \mathbf{k}) &= \chi_+^\dagger(\mathbf{k}') \hat{\mathbf{V}}(\mathbf{k}', \mathbf{k}) \chi_+(\mathbf{k}), \\
 \bar{A}_{++}^0(\mathbf{k}', \mathbf{k}) &= \chi_+^\dagger(\mathbf{k}') \hat{A}^0(\mathbf{k}', \mathbf{k}) \chi_+(\mathbf{k}), \\
 \bar{\mathbf{A}}_{++}(\mathbf{k}', \mathbf{k}) &= \chi_+^\dagger(\mathbf{k}') \hat{\mathbf{A}}(\mathbf{k}', \mathbf{k}) \chi_+(\mathbf{k}).
 \end{aligned}$$

In terms of the observed scattering angles θ and φ related to the outgoing (anti)neutrino three-momentum $\mathbf{k}' \equiv \mathbf{k}_f$, the matrices of the vector $\hat{\nu}^{0,1,2,3}(\mathbf{k}', \mathbf{k})$

³⁰A possibility of a change in (anti)neutrino helicity after the interaction is not considered. Though, from the point of view of searching for manifestations of New Physics, this effect may in principle have a right to exist if, for example, one assumes non-zero (anti)neutrino masses.

and axial-vector $\hat{A}^{0,1,2,3}(\mathbf{k}', \mathbf{k})$ leptonic currents defined by (260) and (261) have the following explicit form:

$$\begin{aligned}
 V^0(k_f) &\equiv \hat{V}^0(\mathbf{k}', \mathbf{k}) = \lambda^2[1 + \alpha_k^\dagger \alpha_k] \\
 &= E_v \begin{pmatrix} 1 + \cos \theta & -e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & 1 + \cos \theta \end{pmatrix}, \\
 V^1(k_f) &\equiv \hat{V}^1(\mathbf{k}', \mathbf{k}) = \lambda^2[\alpha_k \sigma_1 + \sigma_1 \alpha_k] \\
 &= E_v \begin{pmatrix} e^{-i\phi} \sin \theta & -1 + \cos \theta (112) \\ 1 - \cos \theta & e^{i\phi} \sin \theta (113) \end{pmatrix}, \\
 V^2(k_f) &\equiv \hat{V}^2(\mathbf{k}', \mathbf{k}) = \lambda^2[\alpha_k \sigma_2 + \sigma_2 \alpha_k] \\
 &= E_v \begin{pmatrix} ie^{-i\phi} \sin \theta & i - i \cos \theta \\ i - i \cos \theta & -ie^{i\phi} \sin \theta \end{pmatrix} = iA^1(k_f), \\
 V^3(k_f) &\equiv \hat{V}^3(\mathbf{k}', \mathbf{k}) = \lambda^2[\alpha_k \sigma_3 + \sigma_3 \alpha_k] \\
 &= E_v \begin{pmatrix} 1 + \cos \theta & -e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & 1 + \cos \theta \end{pmatrix} = V^0(k_f);
 \end{aligned} \tag{262}$$

$$\begin{aligned}
 A^0(k_f) &\equiv \hat{A}^0(\mathbf{k}', \mathbf{k}) = \lambda^2[\alpha_{k'} + \alpha_k] \\
 &= E_v \begin{pmatrix} 1 + \cos \theta & e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & -1 - \cos \theta \end{pmatrix}, \\
 A^1(k_f) &\equiv \hat{A}^1(\mathbf{k}', \mathbf{k}) = \lambda^2[\sigma_1 + \alpha_{k'} \sigma_1 \alpha_k] \\
 &= E_v \begin{pmatrix} e^{-i\phi} \sin \theta & 1 - \cos \theta \\ 1 - \cos \theta & -e^{i\phi} \sin \theta \end{pmatrix}, \\
 A^2(k_f) &\equiv \hat{A}^2(\mathbf{k}', \mathbf{k}) = \lambda^2[\sigma_2 + \alpha_{k'} \sigma_2 \alpha_k] \\
 &= E_v \begin{pmatrix} ie^{-i\phi} \sin \theta & -i(1 - \cos \theta) \\ i(1 - \cos \theta) & ie^{i\phi} \sin \theta \end{pmatrix} = iV^1(k_f), \\
 A^3(k_f) &\equiv \hat{A}^3(\mathbf{k}', \mathbf{k}) = \lambda^2[\sigma_3 + \alpha_{k'} \sigma_3 \alpha_k] \\
 &= E_v \begin{pmatrix} 1 + \cos \theta & e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & -1 - \cos \theta \end{pmatrix} = A^0(k_f).
 \end{aligned} \tag{263}$$

Indeed, direct calculations show that

$$\begin{aligned}
 V^0(k_f) &= \lambda^2 + \lambda^2 \alpha_{k'} \alpha_k = E_v \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
 &+ E_v \begin{pmatrix} \cos \theta & e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & -\cos \theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
 &= E_v \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + E_v \begin{pmatrix} \cos \theta & -e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & \cos \theta \end{pmatrix} \\
 &= E_v \begin{pmatrix} 1 + \cos \theta & -e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & 1 + \cos \theta \end{pmatrix},
 \end{aligned}$$

$$\begin{aligned}
 V^1(k_f) &= \lambda^2[\alpha_{k'} \sigma_1 + \sigma_1 \alpha_k] \\
 &= E_v \left[\begin{pmatrix} \cos \theta & e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & -\cos \theta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] \\
 &= E_v \left[\begin{pmatrix} e^{-i\phi} \sin \theta & \cos \theta \\ -\cos \theta & e^{i\phi} \sin \theta \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right] \\
 &\times E_v \begin{pmatrix} e^{-i\phi} \sin \theta & -1 + \cos \theta \\ 1 - \cos \theta & e^{i\phi} \sin \theta \end{pmatrix}, \\
 V^2(k_f) &= \lambda^2[\alpha_{k'} \sigma_2 + \sigma_2 \alpha_k] \\
 &= E_v \left[\begin{pmatrix} \cos \theta & e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & -\cos \theta \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] \\
 &= E_v \left[\begin{pmatrix} ie^{-i\phi} \sin \theta & -i \cos \theta \\ -i \cos \theta & -ie^{i\phi} \sin \theta \end{pmatrix} + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right] \\
 &= E_v \begin{pmatrix} ie^{-i\phi} \sin \theta & i - i \cos \theta \\ i - i \cos \theta & -ie^{i\phi} \sin \theta \end{pmatrix}, \\
 V^3(k_f) &= \lambda^2[\alpha_{k'} \sigma_3 + \sigma_3 \alpha_k] \\
 &= E_v \left[\begin{pmatrix} \cos \theta & e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & -\cos \theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] \\
 &= E_v \begin{pmatrix} 1 + \cos \theta & -e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & 1 + \cos \theta \end{pmatrix}.
 \end{aligned}$$

$$\begin{aligned}
 A^0(k_f) &= \lambda^2[\alpha_{k'} + \alpha_k] \\
 &= E_v \left[\begin{pmatrix} \cos \theta & e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & -\cos \theta \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] \\
 &= E_v \begin{pmatrix} 1 + \cos \theta & e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & -1 - \cos \theta \end{pmatrix}, \\
 A^1(k_f) &= \lambda^2 \sigma_1 + \lambda^2 \alpha_{k'} \sigma_1 \alpha_k \\
 &= E_v \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} \cos \theta & e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & -\cos \theta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] \\
 &= E_v \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} \cos \theta & e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & -\cos \theta \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right] \\
 &\times E_v \begin{pmatrix} e^{-i\phi} \sin \theta & 1 - \cos \theta \\ 1 - \cos \theta & -e^{i\phi} \sin \theta \end{pmatrix}, \\
 A^2(k_f) &= \lambda^2 \sigma_2 + \lambda^2 \alpha_{k'} \sigma_2 \alpha_k \\
 &= E_v \begin{pmatrix} ie^{-i\phi} \sin \theta & -i(1 - \cos \theta) \\ i(1 - \cos \theta) & ie^{i\phi} \sin \theta \end{pmatrix}, \\
 A^3(k_f) &= \lambda^2 \sigma_3 + \lambda^2 \alpha_{k'} \sigma_3 \alpha_k \\
 &= E_v \begin{pmatrix} 1 + \cos \theta & e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & -1 - \cos \theta \end{pmatrix}.
 \end{aligned}$$

Using explicit expressions for matrices (262) and (263) and the explicit form of the bispinors

$$\chi_{-}(\mathbf{k}_i) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$\chi_{-}(\mathbf{k}_f) = \cos \frac{\theta}{2} \begin{pmatrix} -e^{-i\phi} \tan \frac{\theta}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} -e^{-i\phi} \sin \frac{\theta}{2} \\ \cos \frac{\theta}{2} \end{pmatrix},$$

one can calculate all components of the neutrino vector and axial-vector currents

$$\begin{aligned} V_{--}^0 &= \chi_{-}^{\dagger}(\mathbf{k}_f) V^0(k_f) \chi_{-}(\mathbf{k}_i) \\ &= \chi_{-}^{\dagger}(\mathbf{k}_f) \begin{pmatrix} 1 + \cos \theta & -e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & 1 + \cos \theta \end{pmatrix} \chi_{-}(\mathbf{k}_i) \\ &= \cos \frac{\theta}{2} \begin{pmatrix} -e^{-i\phi} \tan \frac{\theta}{2} \\ 1 \end{pmatrix}^{\dagger} \begin{pmatrix} 1 + \cos \theta & -e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & 1 + \cos \theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \cos \frac{\theta}{2} \begin{pmatrix} -e^{i\phi} \tan \frac{\theta}{2}, 1 \end{pmatrix} \begin{pmatrix} -e^{-i\phi} \sin \theta \\ 1 + \cos \theta \end{pmatrix} \\ &= \cos \frac{\theta}{2} \left(1 + \sin \theta \tan \frac{\theta}{2} + \cos \theta \right) = 2 \cos \frac{\theta}{2}; \\ V_{--}^1 &= \chi_{-}^{\dagger}(\mathbf{k}_f) V^1(k_f) \chi_{-}(\mathbf{k}_i) = \cos \frac{\theta}{2} \begin{pmatrix} -e^{-i\phi} \tan \frac{\theta}{2} \\ 1 \end{pmatrix}^{\dagger} \\ &\quad \times \begin{pmatrix} e^{-i\phi} \sin \theta & -1 + \cos \theta \\ 1 - \cos \theta & e^{i\phi} \sin \theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \cos \frac{\theta}{2} \begin{pmatrix} -e^{i\phi} \tan \frac{\theta}{2}, 1 \end{pmatrix} \begin{pmatrix} -1 + \cos \theta \\ e^{i\phi} \sin \theta \end{pmatrix} \\ &= \cos \frac{\theta}{2} e^{i\phi} \left[\tan \frac{\theta}{2} (1 - \cos \theta) + \sin \theta \right] = 2e^{i\phi} \sin \frac{\theta}{2}; \\ V_{--}^2 &= \chi_{-}^{\dagger}(\mathbf{k}_f) V^2(k_f) \chi_{-}(\mathbf{k}_i) = \cos \frac{\theta}{2} \begin{pmatrix} -e^{-i\phi} \tan \frac{\theta}{2} \\ 1 \end{pmatrix}^{\dagger} \\ &\quad \times \begin{pmatrix} ie^{-i\phi} \sin \theta & i - i \cos \theta \\ i - i \cos \theta & -ie^{i\phi} \sin \theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \cos \frac{\theta}{2} i \begin{pmatrix} -e^{i\phi} \tan \frac{\theta}{2}, 1 \end{pmatrix} \begin{pmatrix} 1 - \cos \theta \\ -e^{i\phi} \sin \theta \end{pmatrix} \\ &= -ie^{i\phi} \cos \frac{\theta}{2} \left[\tan \frac{\theta}{2} (1 - \cos \theta) + \sin \theta \right] = -i2e^{i\phi} \sin \frac{\theta}{2}; \\ V_{--}^3 &= \chi_{-}^{\dagger}(\mathbf{k}_f) V^3(k_f) \chi_{-}(\mathbf{k}_i) = \cos \frac{\theta}{2} \begin{pmatrix} -e^{-i\phi} \tan \frac{\theta}{2} \\ 1 \end{pmatrix}^{\dagger} \\ &\quad \times \begin{pmatrix} 1 + \cos \theta & -e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & 1 + \cos \theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \cos \frac{\theta}{2} \begin{pmatrix} -e^{i\phi} \tan \frac{\theta}{2}, 1 \end{pmatrix} \\ &\quad \times \begin{pmatrix} -e^{-i\phi} \sin \theta \\ 1 + \cos \theta \end{pmatrix} \cos \frac{\theta}{2} \left(1 + \cos \theta + \tan \frac{\theta}{2} \sin \theta \right) = 2 \cos \frac{\theta}{2}. \end{aligned}$$

$$\begin{aligned} A_{--}^0 &= \chi_{-}^{\dagger}(\mathbf{k}_f) A^0(k_f) \chi_{-}(\mathbf{k}_i) = \cos \frac{\theta}{2} \begin{pmatrix} -e^{-i\phi} \tan \frac{\theta}{2} \\ 1 \end{pmatrix}^{\dagger} \\ &\quad \times \begin{pmatrix} 1 + \cos \theta & e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & -1 - \cos \theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \cos \frac{\theta}{2} \begin{pmatrix} -e^{i\phi} \tan \frac{\theta}{2}, 1 \end{pmatrix} \begin{pmatrix} e^{-i\phi} \sin \theta \\ -1 - \cos \theta \end{pmatrix} \\ &= -\cos \frac{\theta}{2} [1 + \cos \theta + \tan \frac{\theta}{2} \sin \theta] = -2 \cos \frac{\theta}{2}; \end{aligned}$$

$$\begin{aligned} A_{--}^1 &= \chi_{-}^{\dagger}(\mathbf{k}_f) A^1(k_f) \chi_{-}(\mathbf{k}_i) = \cos \frac{\theta}{2} \begin{pmatrix} -e^{-i\phi} \tan \frac{\theta}{2} \\ 1 \end{pmatrix}^{\dagger} \\ &\quad \times \begin{pmatrix} e^{-i\phi} \sin \theta & 1 - \cos \theta \\ 1 - \cos \theta & -e^{i\phi} \sin \theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \cos \frac{\theta}{2} \begin{pmatrix} -e^{i\phi} \tan \frac{\theta}{2}, 1 \end{pmatrix} \\ &\quad \times \begin{pmatrix} 1 - \cos \theta \\ -e^{i\phi} \sin \theta \end{pmatrix} = -e^{i\phi} \cos \frac{\theta}{2} \\ &\quad \times [\tan \frac{\theta}{2} (1 - \cos \theta) + \sin \theta] = -2e^{i\phi} \sin \frac{\theta}{2}; \end{aligned}$$

$$\begin{aligned} A_{--}^2 &= \chi_{-}^{\dagger}(\mathbf{k}_f) A^2(k_f) \chi_{-}(\mathbf{k}_i) = \cos \frac{\theta}{2} \begin{pmatrix} -e^{-i\phi} \tan \frac{\theta}{2} \\ 1 \end{pmatrix}^{\dagger} \\ &\quad \times \begin{pmatrix} ie^{-i\phi} \sin \theta & -i(1 - \cos \theta) \\ i(1 - \cos \theta) & ie^{i\phi} \sin \theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \cos \frac{\theta}{2} \begin{pmatrix} -e^{i\phi} \tan \frac{\theta}{2}, 1 \end{pmatrix} \begin{pmatrix} -i(1 - \cos \theta) \\ ie^{i\phi} \sin \theta \end{pmatrix} \\ &= ie^{i\phi} \cos \frac{\theta}{2} [\tan \frac{\theta}{2} (1 - \cos \theta) + \sin \theta] = i2e^{i\phi} \sin \frac{\theta}{2}; \end{aligned}$$

$$\begin{aligned} A_{--}^3 &= \chi_{-}^{\dagger}(\mathbf{k}_f) A^3(k_f) \chi_{-}(\mathbf{k}_i) = \cos \frac{\theta}{2} \begin{pmatrix} -e^{-i\phi} \tan \frac{\theta}{2} \\ 1 \end{pmatrix}^{\dagger} \\ &\quad \times \begin{pmatrix} 1 + \cos \theta & e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & -1 - \cos \theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \cos \frac{\theta}{2} \begin{pmatrix} -e^{i\phi} \tan \frac{\theta}{2}, 1 \end{pmatrix} \begin{pmatrix} e^{-i\phi} \sin \theta \\ -1 - \cos \theta \end{pmatrix} = -2 \cos \frac{\theta}{2}. \end{aligned}$$

In these calculations, the common factor E_{ν} was temporarily omitted. With it returned to its place, for the components of the neutrino vector weak current in the neutrino and nucleon center-of-mass system in the helicity basis we have

$$\begin{aligned} V_{--}^0(\mathbf{k}', \mathbf{k}) &= \bar{u}(\mathbf{k}', -1) \gamma^0 u(\mathbf{k}, -1) \\ &= \chi_{-}^{\dagger}(\mathbf{k}_f) V^0(k_f) \chi_{-}(\mathbf{k}_i) = 2E_{\nu} \cos \frac{\theta}{2}, \\ \mathbf{V}_{--}^1(\mathbf{k}', \mathbf{k}) &= \bar{u}(\mathbf{k}', -1) \gamma^1 u(\mathbf{k}, -1) \\ &= \chi_{-}^{\dagger}(\mathbf{k}_f) V^1(k_f) \chi_{-}(\mathbf{k}_i) = 2e^{i\phi} E_{\nu} \sin \frac{\theta}{2}, \end{aligned}$$

$$\begin{aligned}
V_{--}^2(\mathbf{k}', \mathbf{k}) &= \bar{u}(\mathbf{k}', -1)\gamma^2 u(\mathbf{k}, -1) \\
&= \chi_{-}^{\dagger}(\mathbf{k}_f) V^2(k_f) \chi_{-}(\mathbf{k}_i) = -2ie^{i\phi} E_v \sin \frac{\theta}{2}, = -V_{--}^1(\mathbf{k}', \mathbf{k}), \\
V_{--}^3(\mathbf{k}', \mathbf{k}) &= \bar{u}(\mathbf{k}', -1)\gamma^3 u(\mathbf{k}, -1) \\
&= \chi_{-}^{\dagger}(\mathbf{k}_f) V^3(k_f) \chi_{-}(\mathbf{k}_i) = 2E_v \cos \frac{\theta}{2} = V_{--}^0(\mathbf{k}', \mathbf{k}).
\end{aligned}$$

For the components of the neutrino axial weak current we correspondingly have

$$\begin{aligned}
A_{--}^0(\mathbf{k}', \mathbf{k}) &= \bar{u}(\mathbf{k}', -1)\gamma^0 \gamma_5 u(\mathbf{k}, -1) = \chi_{-}^{\dagger}(\mathbf{k}_f) A^0(k_f) \chi_{-}(\mathbf{k}_i) \\
&= -2E_v \cos \frac{\theta}{2} = -V_{--}^0(\mathbf{k}', \mathbf{k}), \\
A_{--}^1(\mathbf{k}', \mathbf{k}) &= \bar{u}(\mathbf{k}', -1)\gamma^1 \gamma_5 u(\mathbf{k}, -1) = \chi_{-}^{\dagger}(\mathbf{k}_f) A^1(k_f) \chi_{-}(\mathbf{k}_i) \\
&= -2e^{i\phi} E_v \sin \frac{\theta}{2} = -V_{--}^1(\mathbf{k}', \mathbf{k}), \\
A_{--}^2(\mathbf{k}', \mathbf{k}) &= \bar{u}(\mathbf{k}', -1)\gamma^2 \gamma_5 u(\mathbf{k}, -1) = \chi_{-}^{\dagger}(\mathbf{k}_f) A^2(k_f) \chi_{-}(\mathbf{k}_i) \\
&= 2ie^{i\phi} E_v \sin \frac{\theta}{2} = -A_{--}^1(\mathbf{k}', \mathbf{k}), \\
A_{--}^3(\mathbf{k}', \mathbf{k}) &= \bar{u}(\mathbf{k}', -1)\gamma^3 \gamma_5 u(\mathbf{k}, -1) = \chi_{-}^{\dagger}(\mathbf{k}_f) A^3(k_f) \chi_{-}(\mathbf{k}_i) \\
&= -2E_v \cos \frac{\theta}{2} = A_{--}^0(\mathbf{k}', \mathbf{k}).
\end{aligned}$$

In a more convenient and compact form, neutrino currents are written as

$$\begin{aligned}
V_{--}^{\lambda}(\mathbf{k}_f) &= 2E_v \left(\cos \frac{\theta}{2}, e^{i\phi} \sin \frac{\theta}{2}, -ie^{i\phi} \sin \frac{\theta}{2}, \cos \frac{\theta}{2} \right), \\
A_{--}^{\lambda}(\mathbf{k}_f) &= 2E_v \left(-\cos \frac{\theta}{2}, -e^{i\phi} \sin \frac{\theta}{2}, ie^{i\phi} \sin \frac{\theta}{2}, -\cos \frac{\theta}{2} \right) \\
&= -V_{--}^{\lambda}(\mathbf{k}_f).
\end{aligned} \tag{264}$$

In [19], considering the designations $c_{\theta/2} \equiv \cos \frac{\theta}{2}$, $s_{\theta/2} \equiv \sin \frac{\theta}{2}$, these expressions are

$$\begin{aligned}
V_{--}^{\chi(0,1,2,3)} &= 2E_v (c_{\theta/2}, s_{\theta/2} e^{+i\phi}, -is_{\theta/2} e^{+i\phi}, c_{\theta/2}), \\
A_{--}^{\chi(0,1,2,3)} &= 2E_v (-c_{\theta/2}, -s_{\theta/2} e^{+i\phi}, is_{\theta/2} e^{+i\phi}, -c_{\theta/2}).
\end{aligned} \tag{265}$$

From (264) it follows that, first, the neutrino right-handed weak current is identically equal to zero, $V_{--}^{\lambda} + A_{--}^{\lambda} \equiv 0$, and, second, the left-handed $V - A$ current has the form

$$\begin{aligned}
l_v(\mathbf{k}', \mathbf{k}) &= \bar{u}(\mathbf{k}', -1)\gamma^{\mu} u(\mathbf{k}, -1) - \bar{u}(\mathbf{k}', -1) \\
&\times \gamma^{\mu} \gamma_5 u(\mathbf{k}, -1) = \bar{u}(\mathbf{k}', -1)\gamma^{\mu} (1 - \gamma_5) u(\mathbf{k}, -1) \\
&= V_{--}^{\lambda}(\mathbf{k}_f) - A_{--}^{\lambda}(\mathbf{k}_f) \\
&\times 4E_v \left(\cos \frac{\theta}{2}, e^{i\phi} \sin \frac{\theta}{2}, -ie^{i\phi} \sin \frac{\theta}{2}, \cos \frac{\theta}{2} \right).
\end{aligned} \tag{266}$$

This neutrino weak current should be convoluted with the nucleon weak current, similarly obtained

below, for finding the scalar product $(l_v^{\lambda}, h^{\lambda})$ in the helicity basis.

Now we turn to calculation of the components of the vector $\bar{V}_{++}^{0,1,2,3}(\mathbf{k}', \mathbf{k}) = \chi_{+}^{\dagger}(\mathbf{k}') \hat{V}^{0,1,2,3}(k_f) \chi_{+}(\mathbf{k})$ and axial $\bar{A}_{++}^{0,1,2,3}(\mathbf{k}', \mathbf{k}) = \chi_{+}^{\dagger}(\mathbf{k}') \hat{A}^{0,1,2,3}(k_f) \chi_{+}(\mathbf{k})$ antineutrino currents. Similar to the “neutrino” case, expressions for matrices (262) and (263) and the explicit form of “antineutrino” spinors

$$\begin{aligned}
\chi_{+}(\mathbf{k}_i) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_{+}(\mathbf{k}_f) = e^{i\phi} \sin \frac{\theta}{2} \begin{pmatrix} e^{-i\phi} \cot \frac{\theta}{2} \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos \frac{\theta}{2} \\ e^{i\phi} \sin \frac{\theta}{2} \end{pmatrix},
\end{aligned}$$

are used to calculate all components of the antineutrino vector and axial-vector currents

$$\begin{aligned}
V_{++}^0 &= \chi_{+}^{\dagger}(\mathbf{k}_f) V^0(k_f) \chi_{+}(\mathbf{k}_i) \\
&= \chi_{+}^{\dagger}(\mathbf{k}_f) \begin{pmatrix} 1 + \cos \theta & -e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & 1 + \cos \theta \end{pmatrix} \chi_{+}(\mathbf{k}_i) \\
&= e^{-i\phi} \sin \frac{\theta}{2} \begin{pmatrix} e^{-i\phi} \cot \frac{\theta}{2} \\ 1 \end{pmatrix}^{\dagger} \begin{pmatrix} 1 + \cos \theta & -e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & 1 + \cos \theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= e^{-i\phi} \sin \frac{\theta}{2} \begin{pmatrix} e^{i\phi} \cot \frac{\theta}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 + \cos \theta \\ e^{i\phi} \sin \theta \end{pmatrix} \\
&= e^{-i\phi} \sin \frac{\theta}{2} e^{i\phi} [\cot \frac{\theta}{2} (1 + \cos \theta) + \sin \theta] = 2 \cos \frac{\theta}{2}; \\
V_{++}^1 &= \chi_{+}^{\dagger}(\mathbf{k}_f) V^1(k_f) \chi_{+}(\mathbf{k}_i) = e^{-i\phi} \sin \frac{\theta}{2} \begin{pmatrix} e^{-i\phi} \cot \frac{\theta}{2} \\ 1 \end{pmatrix}^{\dagger} \\
&\times \begin{pmatrix} e^{-i\phi} \sin \theta & -1 + \cos \theta \\ 1 - \cos \theta & e^{i\phi} \sin \theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = e^{-i\phi} \sin \frac{\theta}{2} \begin{pmatrix} e^{i\phi} \cot \frac{\theta}{2} & 1 \end{pmatrix} \\
&\times \begin{pmatrix} e^{-i\phi} \sin \theta \\ 1 - \cos \theta \end{pmatrix} e^{-i\phi} \sin \frac{\theta}{2} [\cot \frac{\theta}{2} \sin \theta + 1 - \cos \theta] \\
&= 2e^{-i\phi} \sin \frac{\theta}{2}; \\
V_{++}^2 &= \chi_{+}^{\dagger}(\mathbf{k}_f) V^2(k_f) \chi_{+}(\mathbf{k}_i) = e^{-i\phi} \sin \frac{\theta}{2} \begin{pmatrix} e^{-i\phi} \cot \frac{\theta}{2} \\ 1 \end{pmatrix}^{\dagger} \\
&\times \begin{pmatrix} ie^{-i\phi} \sin \theta & i - i \cos \theta \\ i - i \cos \theta & -ie^{i\phi} \sin \theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = e^{-i\phi} \sin \frac{\theta}{2} \begin{pmatrix} e^{i\phi} \cot \frac{\theta}{2} & 1 \end{pmatrix} \\
&\times \begin{pmatrix} ie^{-i\phi} \sin \theta \\ i - i \cos \theta \end{pmatrix} ie^{-i\phi} \sin \frac{\theta}{2} \left[\cot \frac{\theta}{2} \sin \theta + 1 - \cos \theta \right] \\
&= 2ie^{-i\phi} \sin \frac{\theta}{2};
\end{aligned}$$

$$\begin{aligned}
V_{++}^3 &= \chi_+^\dagger(\mathbf{k}_f) V^3(k_f) \chi_+(\mathbf{k}_i) \\
&= e^{-i\phi} \sin \frac{\theta}{2} \begin{pmatrix} e^{-i\phi} \cot \frac{\theta}{2} & 1 \\ 1 & \end{pmatrix}^\dagger \begin{pmatrix} 1 + \cos \theta & -e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & 1 + \cos \theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= e^{-i\phi} \sin \frac{\theta}{2} \begin{pmatrix} e^{i\phi} \cot \frac{\theta}{2} & 1 \\ e^{i\phi} \sin \theta & \end{pmatrix} \begin{pmatrix} 1 + \cos \theta \\ e^{i\phi} \sin \theta \end{pmatrix} = 2 \cos \frac{\theta}{2}. \\
A_{++}^0 &= \chi_+^\dagger(\mathbf{k}_f) A^0(k_f) \chi_+(\mathbf{k}_i) = e^{-i\phi} \sin \frac{\theta}{2} \begin{pmatrix} e^{-i\phi} \cot \frac{\theta}{2} & 1 \\ 1 & \end{pmatrix}^\dagger \\
&\quad \times \begin{pmatrix} 1 + \cos \theta & e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & -1 - \cos \theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= e^{-i\phi} \sin \frac{\theta}{2} \begin{pmatrix} e^{i\phi} \cot \frac{\theta}{2} & 1 \\ e^{i\phi} \sin \theta & \end{pmatrix} \begin{pmatrix} 1 + \cos \theta \\ e^{i\phi} \sin \theta \end{pmatrix} = 2 \cos \frac{\theta}{2}; \\
A_{++}^1 &= \chi_+^\dagger(\mathbf{k}_f) A^1(k_f) \chi_+(\mathbf{k}_i) E_v = e^{-i\phi} \sin \frac{\theta}{2} \begin{pmatrix} e^{-i\phi} \cot \frac{\theta}{2} & 1 \\ 1 & \end{pmatrix}^\dagger \\
&\quad \times \begin{pmatrix} e^{-i\phi} \sin \theta & 1 - \cos \theta \\ 1 - \cos \theta & -e^{i\phi} \sin \theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= e^{-i\phi} \sin \frac{\theta}{2} \begin{pmatrix} e^{i\phi} \cot \frac{\theta}{2} & 1 \\ e^{i\phi} \sin \theta & \end{pmatrix} \begin{pmatrix} e^{-i\phi} \sin \theta \\ 1 - \cos \theta \end{pmatrix} = 2e^{-i\phi} \sin \frac{\theta}{2}; \\
A_{++}^2 &= \chi_+^\dagger(\mathbf{k}_f) A^2(k_f) \chi_+(\mathbf{k}_i) = e^{-i\phi} \sin \frac{\theta}{2} \begin{pmatrix} e^{-i\phi} \cot \frac{\theta}{2} & 1 \\ 1 & \end{pmatrix}^\dagger \\
&\quad \times \begin{pmatrix} ie^{-i\phi} \sin \theta & -i(1 - \cos \theta) \\ i(1 - \cos \theta) & ie^{i\phi} \sin \theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= e^{-i\phi} \sin \frac{\theta}{2} \begin{pmatrix} e^{i\phi} \cot \frac{\theta}{2} & 1 \\ e^{i\phi} \sin \theta & \end{pmatrix} \begin{pmatrix} ie^{-i\phi} \sin \theta \\ i(1 - \cos \theta) \end{pmatrix} = 2ie^{-i\phi} \sin \frac{\theta}{2}; \\
A_{++}^3 &= \chi_+^\dagger(\mathbf{k}_f) A^3(k_f) \chi_+(\mathbf{k}_i) e^{-i\phi} \sin \frac{\theta}{2} \begin{pmatrix} e^{-i\phi} \cot \frac{\theta}{2} & 1 \\ 1 & \end{pmatrix}^\dagger \\
&\quad \times \begin{pmatrix} 1 + \cos \theta & e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & -1 - \cos \theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= e^{-i\phi} \sin \frac{\theta}{2} \begin{pmatrix} e^{i\phi} \cot \frac{\theta}{2} & 1 \\ e^{i\phi} \sin \theta & \end{pmatrix} \begin{pmatrix} 1 + \cos \theta \\ e^{i\phi} \sin \theta \end{pmatrix} = 2 \cos \frac{\theta}{2}.
\end{aligned}$$

As a result (with factor E_v restored), components of the antineutrino vector and axial-vector currents in the helicity basis have the form

$$\begin{aligned}
\bar{V}_{++}^{\chi(0,1,2,3)}(\mathbf{k}_f) &= 2E_v e^{-i\phi} \\
&\quad \times \left(e^{i\phi} \cos \frac{\theta}{2}, \sin \frac{\theta}{2}, i \sin \frac{\theta}{2}, e^{i\phi} \cos \frac{\theta}{2} \right) \quad (267) \\
&= 2E_v \left(\cos \frac{\theta}{2}, e^{-i\phi} \sin \frac{\theta}{2}, ie^{-i\phi} \sin \frac{\theta}{2}, \cos \frac{\theta}{2} \right),
\end{aligned}$$

$$\begin{aligned}
\bar{A}_{++}^{\chi(0,1,2,3)}(\mathbf{k}_f) &= 2E_v e^{-i\phi} \\
&\quad \times \left(e^{i\phi} \cos \frac{\theta}{2}, \sin \frac{\theta}{2}, i \sin \frac{\theta}{2}, e^{i\phi} \cos \frac{\theta}{2} \right) \\
&= \bar{V}_{++}^{\chi(0,1,2,3)}(\mathbf{k}_f).
\end{aligned}$$

From the results for the components of antineutrino weak currents (267) it follows that for the antineutrino, by total contrast to the neutrino case (266), first, the difference of the vector and axial-vector currents is zero, $\bar{V}_{++}^\chi - \bar{A}_{++}^\chi \equiv 0$, and, second, the right-handed weak current has the form

$$\begin{aligned}
l_v(\mathbf{k}', \mathbf{k}) &= \bar{v}(\mathbf{k}', +1) \gamma^\mu v(\mathbf{k}, +1) \\
&\quad + \bar{v}(\mathbf{k}', +1) \gamma^\mu \gamma_5 v(\mathbf{k}, +1) \\
&= \bar{v}(\mathbf{k}', +1) \gamma^\mu (1 + \gamma_5) v(\mathbf{k}, +1) \quad (268) \\
&= \bar{V}_{++}^\chi(\mathbf{k}_f) + \bar{A}_{++}^\chi(\mathbf{k}_f) 4E_v \\
&\quad \times \left(\cos \frac{\theta}{2}, e^{-i\phi} \sin \frac{\theta}{2}, ie^{-i\phi} \sin \frac{\theta}{2}, \cos \frac{\theta}{2} \right).
\end{aligned}$$

Thus, at the form of the V - and A -weak interaction specified by formulas (251), the relation $\bar{V}_{++}^\chi - \bar{A}_{++}^\chi = 0$ means that if the spin of the massless fermion is parallel to its momentum (case $[++]$, i.e., v_R and $\bar{v}_R$), this fermion cannot participate in the weak interaction of the $V-A$ type. However, it can take part in the weak interaction of the $V+A$ type, because $\bar{V}_{++}^\chi + \bar{A}_{++}^\chi \neq 0$, which the really existing right-handed antineutrino $\bar{v}_R$ actually does, unlike the nonexistent right-handed neutrino v_R . When the spin is antiparallel to the momentum (case $[- -]$, i.e., v_L and $\bar{v}_L$), the situation is reverse, i.e., the left-handed (real) neutrino v_L participates in the $V-A$ -interaction, and the left-handed antineutrino $\bar{v}_L$ is not manifested in it, because this antineutrino does not exist.

The above antineutrino weak current should be convolved with the similar nucleon current to obtain the desired scalar product (l_v, h^χ) .

Helicity basis: Components of nucleon currents

Based on formulas (251) and (252) for the vector and axial-vector nucleon current

$$\begin{aligned}
V_{r'r}^0(\mathbf{p}', \mathbf{p}) &= \bar{u}(\mathbf{p}', r') \gamma_0 u(\mathbf{p}, r) = \chi_{r'}^\dagger(\mathbf{p}') \hat{V}^0(\mathbf{p}_f) \chi_r(\mathbf{p}), \\
&\quad \text{where } \hat{V}^0(\mathbf{p}_f) \equiv \lambda_+^2 + \lambda_-^2 \alpha_p \alpha_p;
\end{aligned}$$

$$\begin{aligned}
V_{r'r}^1(\mathbf{p}', \mathbf{p}) &= \bar{u}(\mathbf{p}', r') \gamma_1 u(\mathbf{p}, r) = \chi_{r'}^\dagger(\mathbf{p}') \hat{V}^1(\mathbf{p}_f) \chi_r(\mathbf{p}), \\
&\quad \text{where } \hat{V}^1(\mathbf{p}_f) \equiv \lambda_- \lambda_+ [\alpha_p \sigma + \sigma \alpha_p];
\end{aligned}$$

$$\begin{aligned}
A_{r'r}^0(\mathbf{p}', \mathbf{p}) &= \bar{u}(\mathbf{p}', r') \gamma_0 \gamma_5 u(\mathbf{p}, r) = \chi_{r'}^\dagger(\mathbf{p}') \hat{A}^0(\mathbf{p}_f) \chi_r(\mathbf{p}), \\
&\quad \text{where } \hat{A}^0(\mathbf{p}_f) \equiv \lambda_+ \lambda_- [\alpha_p + \alpha_p];
\end{aligned}$$

$$\begin{aligned}
A_{r'r}^1(\mathbf{p}', \mathbf{p}) &= \bar{u}(\mathbf{p}', r') \gamma_1 \gamma_5 u(\mathbf{p}, r) = \chi_{r'}^\dagger(\mathbf{p}') \hat{A}^1(\mathbf{p}_f) \chi_r(\mathbf{p}), \\
&\quad \text{where } \hat{A}^1(\mathbf{p}_f) \equiv \lambda_+^2 \sigma + \lambda_-^2 \alpha_p \sigma \alpha_p,
\end{aligned}$$

we will calculate components of these currents taking into consideration that the following is valid in the lepton and nucleon center-of-mass system

$$\lambda_{\pm}^2 = E_p \pm m \quad \text{and} \quad \lambda_+ \lambda_- = \sqrt{E_p^2 - m^2} = E_v.$$

First, we obtain explicit expressions for the matrices $\hat{V}^\mu(\mathbf{p}_f)$ and $\hat{A}^\mu(\mathbf{p}_f)$, bearing in mind that, according to (241) and (244),

$$\alpha_p = \mathbf{n}_{p_i} \cdot \boldsymbol{\sigma} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{and } \alpha_{p'} = \mathbf{n}_{p_f} \cdot \boldsymbol{\sigma} = \begin{pmatrix} -\cos \theta & -e^{-i\phi} \sin \theta \\ -e^{i\phi} \sin \theta & \cos \theta \end{pmatrix}.$$

Thus, direct calculations for the matrices of the nucleon vector current yield

$$\begin{aligned} \hat{V}^0(\mathbf{p}_f) &= \lambda_+^2 + \lambda_-^2 \alpha_p \alpha_p = \lambda_+^2 + \lambda_-^2 \begin{pmatrix} -\cos \theta & -e^{-i\phi} \sin \theta \\ -e^{i\phi} \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda_+^2 & 0 \\ 0 & \lambda_+^2 \end{pmatrix} + \begin{pmatrix} \lambda_-^2 \cos \theta & -\lambda_-^2 e^{-i\phi} \sin \theta \\ \lambda_-^2 e^{i\phi} \sin \theta & \lambda_-^2 \cos \theta \end{pmatrix} \\ &= \begin{pmatrix} \lambda_+^2 + \lambda_-^2 \cos \theta & -\lambda_-^2 e^{-i\phi} \sin \theta \\ \lambda_-^2 e^{i\phi} \sin \theta & \lambda_+^2 + \lambda_-^2 \cos \theta \end{pmatrix} = \begin{pmatrix} E_p + m + (E_p - m) \cos \theta & -e^{-i\phi} (E_p - m) \sin \theta \\ e^{i\phi} (E_p - m) \sin \theta & E_p + m + (E_p - m) \cos \theta \end{pmatrix}, \\ \hat{V}^1(\mathbf{p}_f) &= \lambda_- \lambda_+ [\alpha_p \sigma_1 + \sigma_1 \alpha_p] = E_v \left[\begin{pmatrix} -\cos \theta & -e^{-i\phi} \sin \theta \\ -e^{i\phi} \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right] \\ &= E_v \left[\begin{pmatrix} -e^{-i\phi} \sin \theta & -\cos \theta \\ \cos \theta & -e^{i\phi} \sin \theta \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right] = E_v \begin{pmatrix} -e^{-i\phi} \sin \theta & 1 - \cos \theta \\ -1 + \cos \theta & -e^{i\phi} \sin \theta \end{pmatrix}, \\ \hat{V}^2(\mathbf{p}_f) &= \lambda_- \lambda_+ [\alpha_p \sigma_2 + \sigma_2 \alpha_p] = E_v \left[\begin{pmatrix} -\cos \theta & -e^{-i\phi} \sin \theta \\ -e^{i\phi} \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right] \\ &= E_v \left[\begin{pmatrix} -ie^{-i\phi} \sin \theta & i \cos \theta \\ i \cos \theta & ie^{i\phi} \sin \theta \end{pmatrix} + \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \right] = i E_v \begin{pmatrix} -e^{-i\phi} \sin \theta & -1 + \cos \theta \\ -1 + \cos \theta & e^{i\phi} \sin \theta \end{pmatrix}, \\ \hat{V}^3(\mathbf{p}_f) &= \lambda_- \lambda_+ [\alpha_p \sigma_3 + \sigma_3 \alpha_p] = E_v \left[\begin{pmatrix} -\cos \theta & -e^{-i\phi} \sin \theta \\ -e^{i\phi} \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right] \\ &= E_v \left[\begin{pmatrix} -\cos \theta & e^{-i\phi} \sin \theta \\ -e^{i\phi} \sin \theta & -\cos \theta \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right] = E_v \begin{pmatrix} -(1 + \cos \theta) & e^{-i\phi} \sin \theta \\ -e^{i\phi} \sin \theta & -(1 + \cos \theta) \end{pmatrix}. \end{aligned}$$

For the matrices of the axial-vector nucleon current, there is

$$\begin{aligned} \hat{A}^0(\mathbf{p}_f) &= \lambda_+ \lambda_- [\alpha_{p'} + \alpha_p] = E_v \left[\begin{pmatrix} -\cos \theta & -e^{-i\phi} \sin \theta \\ -e^{i\phi} \sin \theta & \cos \theta \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right] = E_v \begin{pmatrix} -(1 + \cos \theta) & -e^{-i\phi} \sin \theta \\ -e^{i\phi} \sin \theta & (1 + \cos \theta) \end{pmatrix}, \\ \hat{A}^1(\mathbf{p}_f) &= \lambda_+^2 \sigma_1 + \lambda_-^2 \alpha_p \sigma_1 \alpha_p = \lambda_+^2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \lambda_-^2 \begin{pmatrix} -\cos \theta & -e^{-i\phi} \sin \theta \\ -e^{i\phi} \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \lambda_+^2 \\ \lambda_+^2 & 0 \end{pmatrix} \\ &\quad + \lambda_-^2 \begin{pmatrix} -\cos \theta & -e^{-i\phi} \sin \theta \\ -e^{i\phi} \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \lambda_+^2 \\ \lambda_+^2 & 0 \end{pmatrix} + \lambda_-^2 \begin{pmatrix} e^{-i\phi} \sin \theta & -\cos \theta \\ -\cos \theta & -e^{i\phi} \sin \theta \end{pmatrix} \\ &= \begin{pmatrix} \lambda_-^2 e^{-i\phi} \sin \theta & \lambda_+^2 - \lambda_-^2 \cos \theta \\ \lambda_+^2 - \lambda_-^2 \cos \theta & -\lambda_-^2 e^{i\phi} \sin \theta \end{pmatrix} = \begin{pmatrix} (E_p - m) e^{-i\phi} \sin \theta & (E_p + m) - (E_p - m) \cos \theta \\ (E_p + m) - (E_p - m) \cos \theta & -(E_p - m) e^{i\phi} \sin \theta \end{pmatrix}, \\ \hat{A}^2(\mathbf{p}_f) &= \lambda_+^2 \sigma_2 + \lambda_-^2 \alpha_p \sigma_2 \alpha_p = \lambda_+^2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \lambda_-^2 \begin{pmatrix} -\cos \theta & -e^{-i\phi} \sin \theta \\ -e^{i\phi} \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -i \lambda_+^2 \\ i \lambda_+^2 & 0 \end{pmatrix} + \lambda_-^2 \begin{pmatrix} -\cos \theta & -e^{-i\phi} \sin \theta \\ -e^{i\phi} \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \lambda_+^2 \\ i \lambda_+^2 & 0 \end{pmatrix} + \lambda_-^2 \begin{pmatrix} ie^{-i\phi} \sin \theta & i \cos \theta \\ -i \cos \theta & ie^{i\phi} \sin \theta \end{pmatrix} \\ &= i \begin{pmatrix} \lambda_-^2 e^{-i\phi} \sin \theta & -\lambda_+^2 + \lambda_-^2 \cos \theta \\ \lambda_+^2 - \lambda_-^2 \cos \theta & \lambda_-^2 e^{i\phi} \sin \theta \end{pmatrix} = i \begin{pmatrix} (E_p - m) e^{-i\phi} \sin \theta & -(E_p + m) + (E_p - m) \cos \theta \\ (E_p + m) - (E_p - m) \cos \theta & (E_p - m) e^{i\phi} \sin \theta \end{pmatrix}, \end{aligned}$$

$$\begin{aligned}
\hat{A}^3(\mathbf{p}_f) &= \lambda_+^2 \sigma_3 + \lambda_-^2 \alpha_p \sigma_3 \alpha_p = \lambda_+^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \lambda_-^2 \begin{pmatrix} -\cos \theta & -e^{-i\phi} \sin \theta \\ -e^{i\phi} \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \lambda_+^2 & 0 \\ 0 & -\lambda_+^2 \end{pmatrix} + \lambda_-^2 \begin{pmatrix} -\cos \theta & -e^{-i\phi} \sin \theta \\ -e^{i\phi} \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \lambda_+^2 & 0 \\ 0 & -\lambda_+^2 \end{pmatrix} + \lambda_-^2 \begin{pmatrix} \cos \theta & e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & -\cos \theta \end{pmatrix} \\
&= \begin{pmatrix} \lambda_+^2 + \lambda_-^2 \cos \theta & \lambda_-^2 e^{-i\phi} \sin \theta \\ \lambda_-^2 e^{i\phi} \sin \theta & -\lambda_+^2 - \lambda_-^2 \cos \theta \end{pmatrix} = \begin{pmatrix} (E_p + m) + (E_p - m) \cos \theta & (E_p - m) e^{-i\phi} \sin \theta \\ (E_p - m) e^{i\phi} \sin \theta & -(E_p + m) - (E_p - m) \cos \theta \end{pmatrix}.
\end{aligned}$$

Recall that, according to (246), in the helicity basis the initial $\chi_{\pm}(\mathbf{p}) = \chi_{\pm}(\mathbf{p}_i)$ and final nucleon bispinors $\chi_{\pm}^{\dagger}(\mathbf{p}') = \chi_{\pm}^{\dagger}(\mathbf{p}_f)$ are normalized by the condition $\chi_{\pm}^{\dagger} \chi_{\pm} = 1$ and have the form

$$\begin{aligned}
\chi_{-}(\mathbf{p}_i) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_{-}^{\dagger}(\mathbf{p}_f) = \sin \frac{\theta}{2} \left(e^{+i\phi} \cot \frac{\theta}{2}, 1 \right) = \left(e^{+i\phi} \cos \frac{\theta}{2}, \sin \frac{\theta}{2} \right), \quad \chi_{+}(\mathbf{p}_i) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\
\chi_{+}^{\dagger}(\mathbf{p}_f) &= -e^{-i\phi} \cos \frac{\theta}{2} \left(-e^{+i\phi} \tan \frac{\theta}{2}, 1 \right) = \left(\sin \frac{\theta}{2}, -e^{-i\phi} \cos \frac{\theta}{2} \right).
\end{aligned} \tag{269}$$

Substituting them into the equations for the vector and axial currents (251) and (252) and using the obtained explicit form of the matrices $\hat{V}^{\mu}(\mathbf{p}_f)$ and $\hat{A}^{\mu}(\mathbf{p}_f)$, we obtain, by analogy with the (anti)neutrino case,³¹ the following expressions for all possible components of nucleon weak currents:

$$\begin{aligned}
V_{--}^0(\mathbf{p}', \mathbf{p}) &= \bar{u}(\mathbf{p}', -1) \gamma_0 u(\mathbf{p}, -1) \\
&= \chi_{-}^{\dagger}(\mathbf{p}_f) V_0(p_f) \chi_{-}(\mathbf{p}_i) = 2e^{i\phi} E_p \cos \frac{\theta}{2}, \\
V_{-+}^0(\mathbf{p}', \mathbf{p}) &= \bar{u}(\mathbf{p}', -1) \gamma_0 u(\mathbf{p}, +1) \\
&= \chi_{-}^{\dagger}(\mathbf{p}_f) V_0(p_f) \chi_{+}(\mathbf{p}_i) = 2m \sin \frac{\theta}{2}, \\
V_{+-}^0(\mathbf{p}', \mathbf{p}) &= \bar{u}(\mathbf{p}', +1) \gamma_0 u(\mathbf{p}, -1) \\
&= \chi_{+}^{\dagger}(\mathbf{p}_f) V_0(p_f) \chi_{-}(\mathbf{p}_i) = 2m \sin \frac{\theta}{2}, \\
V_{++}^0(\mathbf{p}', \mathbf{p}) &= \bar{u}(\mathbf{p}', +1) \gamma_0 u(\mathbf{p}, +1) \\
&= \chi_{+}^{\dagger}(\mathbf{p}_f) V_0(p_f) \chi_{+}(\mathbf{p}_i) = -2E_p e^{-i\phi} \cos \frac{\theta}{2}; \\
V_{--}^1(\mathbf{p}', \mathbf{p}) &= \bar{u}(\mathbf{p}', -1) \gamma_1 u(\mathbf{p}, -1) \\
&= \chi_{-}^{\dagger}(\mathbf{p}_f) V_1(p_f) \chi_{-}(\mathbf{p}_i) = -2E_v \sin \frac{\theta}{2}, \\
V_{-+}^1(\mathbf{p}', \mathbf{p}) &= \bar{u}(\mathbf{p}', -1) \gamma_1 u(\mathbf{p}, +1) \\
&= \chi_{-}^{\dagger}(\mathbf{p}_f) V_1(p_f) \chi_{+}(\mathbf{p}_i) = 0, \\
V_{+-}^1(\mathbf{p}', \mathbf{p}) &= \bar{u}(\mathbf{p}', +1) \gamma_1 u(\mathbf{p}, -1) \\
&= \chi_{+}^{\dagger}(\mathbf{p}_f) V_1(p_f) \chi_{-}(\mathbf{p}_i) = 0, \\
V_{++}^1(\mathbf{p}', \mathbf{p}) &= \bar{u}(\mathbf{p}', +1) \gamma_1 u(\mathbf{p}, +1) \\
&= \chi_{+}^{\dagger}(\mathbf{p}_f) V_1(p_f) \chi_{+}(\mathbf{p}_i) = 2E_v \sin \frac{\theta}{2};
\end{aligned}$$

$$\begin{aligned}
V_{--}^2(\mathbf{p}', \mathbf{p}) &= \bar{u}(\mathbf{p}', -1) \gamma_2 u(\mathbf{p}, -1) \\
&= \chi_{-}^{\dagger}(\mathbf{p}_f) V_2(p_f) \chi_{-}(\mathbf{p}_i) = -2iE_v \sin \frac{\theta}{2} \\
V_{-+}^2(\mathbf{p}', \mathbf{p}) &= \bar{u}(\mathbf{p}', -1) \gamma_2 u(\mathbf{p}, +1) \\
&= \chi_{-}^{\dagger}(\mathbf{p}_f) V_2(p_f) \chi_{+}(\mathbf{p}_i) = 0, \\
V_{+-}^2(\mathbf{p}', \mathbf{p}) &= \bar{u}(\mathbf{p}', +1) \gamma_2 u(\mathbf{p}, -1) \\
&= \chi_{+}^{\dagger}(\mathbf{p}_f) V_2(p_f) \chi_{-}(\mathbf{p}_i) = 0, \\
V_{++}^2(\mathbf{p}', \mathbf{p}) &= \bar{u}(\mathbf{p}', +1) \gamma_2 u(\mathbf{p}, +1) \\
&= \chi_{+}^{\dagger}(\mathbf{p}_f) V_2(p_f) \chi_{+}(\mathbf{p}_i) = -2iE_v \sin \frac{\theta}{2};
\end{aligned}$$

$$\begin{aligned}
V_{--}^3(\mathbf{p}', \mathbf{p}) &= \bar{u}(\mathbf{p}', -1) \gamma_3 u(\mathbf{p}, -1) \\
&= \chi_{-}^{\dagger}(\mathbf{p}_f) V_3(p_f) \chi_{-}(\mathbf{p}_i) = -2e^{i\phi} E_v \cos \frac{\theta}{2} \\
V_{-+}^3(\mathbf{p}', \mathbf{p}) &= \bar{u}(\mathbf{p}', -1) \gamma_3 u(\mathbf{p}, +1) \\
&= \chi_{-}^{\dagger}(\mathbf{p}_f) V_3(p_f) \chi_{+}(\mathbf{p}_i) = 0, \\
V_{+-}^3(\mathbf{p}', \mathbf{p}) &= \bar{u}(\mathbf{p}', +1) \gamma_3 u(\mathbf{p}, -1) \\
&= \chi_{+}^{\dagger}(\mathbf{p}_f) V_3(p_f) \chi_{-}(\mathbf{p}_i) = 0, \\
V_{++}^3(\mathbf{p}', \mathbf{p}) &= \bar{u}(\mathbf{p}', +1) \gamma_3 u(\mathbf{p}, +1) \\
&= \chi_{+}^{\dagger}(\mathbf{p}_f) V_3(p_f) \chi_{+}(\mathbf{p}_i) = 2E_v e^{-i\phi} \cos \frac{\theta}{2}.
\end{aligned}$$

$$\begin{aligned}
A_{--}^0(\mathbf{p}', \mathbf{p}) &= \bar{u}(\mathbf{p}', -1) \gamma_0 \gamma_5 u(\mathbf{p}, -1) \\
&= \chi_{-}^{\dagger}(\mathbf{p}_f) A_0(p_f) \chi_{-}(\mathbf{p}_i) = -2e^{i\phi} E_v \cos \frac{\theta}{2}, \\
A_{-+}^0(\mathbf{p}', \mathbf{p}) &= \bar{u}(\mathbf{p}', -1) \gamma_0 \gamma_5 u(\mathbf{p}, +1) \\
&= \chi_{-}^{\dagger}(\mathbf{p}_f) A_0(p_f) \chi_{+}(\mathbf{p}_i) = 0,
\end{aligned}$$

³¹ Or using in parallel the MATHEMATICA package.

$$\begin{aligned}
A_{+-}^0(\mathbf{p}', \mathbf{p}) &= \bar{u}(\mathbf{p}', +1)\gamma_0\gamma_5 u(\mathbf{p}, -1) \\
&= \chi_+^\dagger(\mathbf{p}_f)A_0(p_f)\chi_-(\mathbf{p}_i) = 0, \\
A_{++}^0(\mathbf{p}', \mathbf{p}) &= \bar{u}(\mathbf{p}', +1)\gamma_0\gamma_5 u(\mathbf{p}, +1) \\
&= \chi_+^\dagger(\mathbf{p}_f)A_0(p_f)\chi_+(\mathbf{p}_i) = -2E_\nu e^{-i\varphi} \cos \frac{\theta}{2}; \\
A_{--}^1(\mathbf{p}', \mathbf{p}) &= \bar{u}(\mathbf{p}', -1)\gamma_1\gamma_5 u(\mathbf{p}, -1) \\
&= \chi_-^\dagger(\mathbf{p}_f)A_1(p_f)\chi_-(\mathbf{p}_i) = 2E_p \sin \frac{\theta}{2}, \\
A_{-+}^1(\mathbf{p}', \mathbf{p}) &= \bar{u}(\mathbf{p}', -1)\gamma_1\gamma_5 u(\mathbf{p}, +1) \\
&= \chi_-^\dagger(\mathbf{p}_f)A_1(p_f)\chi_+(\mathbf{p}_i) = 2e^{i\varphi} m \cos \frac{\theta}{2} \\
A_{+-}^1(\mathbf{p}', \mathbf{p}) &= \bar{u}(\mathbf{p}', +1)\gamma_1\gamma_5 u(\mathbf{p}, -1) \\
&= \chi_+^\dagger(\mathbf{p}_f)A_1(p_f)\chi_-(\mathbf{p}_i) = -2me^{-i\varphi} \cos \frac{\theta}{2} \\
A_{++}^1(\mathbf{p}', \mathbf{p}) &= \bar{u}(\mathbf{p}', +1)\gamma_1\gamma_5 u(\mathbf{p}, +1) \\
&= \chi_+^\dagger(\mathbf{p}_f)A_1(p_f)\chi_+(\mathbf{p}_i) = 2E_p \sin \frac{\theta}{2}; \\
A_{+-}^2(\mathbf{p}', \mathbf{p}) &= \bar{u}(\mathbf{p}', -1)\gamma_2\gamma_5 u(\mathbf{p}, -1) \\
&= \chi_-^\dagger(\mathbf{p}_f)A_2(p_f)\chi_-(\mathbf{p}_i) = 2iE_p \sin \frac{\theta}{2}, \\
A_{-+}^2(\mathbf{p}', \mathbf{p}) &= \bar{u}(\mathbf{p}', -1)\gamma_2\gamma_5 u(\mathbf{p}, +1) \\
&= \chi_-^\dagger(\mathbf{p}_f)A_2(p_f)\chi_+(\mathbf{p}_i) = -2ie^{i\varphi} m \cos \frac{\theta}{2}, \\
A_{+-}^2(\mathbf{p}', \mathbf{p}) &= \bar{u}(\mathbf{p}', +1)\gamma_2\gamma_5 u(\mathbf{p}, -1) \\
&= \chi_+^\dagger(\mathbf{p}_f)A_2(p_f)\chi_-(\mathbf{p}_i) = -2ime^{-i\varphi} \cos \frac{\theta}{2}, \\
A_{++}^2(\mathbf{p}', \mathbf{p}) &= \bar{u}(\mathbf{p}', +1)\gamma_2\gamma_5 u(\mathbf{p}, +1) \\
&= \chi_+^\dagger(\mathbf{p}_f)A_2(p_f)\chi_+(\mathbf{p}_i) = -2iE_p \sin \frac{\theta}{2}; \\
A_{--}^3(\mathbf{p}', \mathbf{p}) &= \bar{u}(\mathbf{p}', -1)\gamma_3\gamma_5 u(\mathbf{p}, -1) \\
&= \chi_-^\dagger(\mathbf{p}_f)A_3(p_f)\chi_-(\mathbf{p}_i) = 2e^{i\varphi} E_p \cos \frac{\theta}{2}, \\
A_{-+}^3(\mathbf{p}', \mathbf{p}) &= \bar{u}(\mathbf{p}', -1)\gamma_3\gamma_5 u(\mathbf{p}, +1) \\
&= \chi_-^\dagger(\mathbf{p}_f)A_3(p_f)\chi_+(\mathbf{p}_i) = -2m \sin \frac{\theta}{2}, \\
A_{+-}^3(\mathbf{p}', \mathbf{p}) &= \bar{u}(\mathbf{p}', +1)\gamma_3\gamma_5 u(\mathbf{p}, -1) = \chi_+^\dagger(\mathbf{p}_f)A_3(p_f)\chi_-(\mathbf{p}_i) = 2m \sin \frac{\theta}{2}, \\
A_{++}^3(\mathbf{p}', \mathbf{p}) &= \bar{u}(\mathbf{p}', +1)\gamma_3\gamma_5 u(\mathbf{p}, +1) = \chi_+^\dagger(\mathbf{p}_f)A_3(p_f)\chi_+(\mathbf{p}_i) = 2E_p e^{-i\varphi} \cos \frac{\theta}{2}.
\end{aligned}$$

With these components organized into compact notation, the four-vector and four-axial-vector nucleon currents in the helicity basis take the form³²

$$\begin{aligned}
V_{--}^\chi &= 2\left(e^{i\varphi} E_p \cos \frac{\theta}{2}, -E_\nu \sin \frac{\theta}{2}, -iE_\nu \sin \frac{\theta}{2}, -e^{i\varphi} E_\nu \cos \frac{\theta}{2}\right), \quad V_{-+}^\chi = 2m\left(\sin \frac{\theta}{2}, 0, 0, 0\right) = (-)2m\left(-\sin \frac{\theta}{2}, 0, 0, 0\right), \\
V_{+-}^\chi &= 2m\left(\sin \frac{\theta}{2}, 0, 0, 0\right), \quad V_{++}^\chi = 2\left(-E_p e^{-i\varphi} \cos \frac{\theta}{2}, E_\nu \sin \frac{\theta}{2}, -iE_\nu \sin \frac{\theta}{2}, E_\nu e^{-i\varphi} \cos \frac{\theta}{2}\right) \\
&= (-)2\left(E_p e^{-i\varphi} \cos \frac{\theta}{2}, -E_\nu \sin \frac{\theta}{2}, iE_\nu \sin \frac{\theta}{2}, -E_\nu e^{-i\varphi} \cos \frac{\theta}{2}\right); \\
A_{--}^\chi &= 2\left(-e^{i\varphi} E_\nu \cos \frac{\theta}{2}, E_p \sin \frac{\theta}{2}, iE_p \sin \frac{\theta}{2}, e^{i\varphi} E_p \cos \frac{\theta}{2}\right), \quad A_{-+}^\chi = 2m\left(0, e^{i\varphi} \cos \frac{\theta}{2}, -ie^{i\varphi} \cos \frac{\theta}{2}, -\sin \frac{\theta}{2}\right) \\
&= (-)2m\left(0, -e^{i\varphi} \cos \frac{\theta}{2}, ie^{i\varphi} \cos \frac{\theta}{2}, \sin \frac{\theta}{2}\right), \quad A_{+-}^\chi = 2m\left(0, -e^{-i\varphi} \cos \frac{\theta}{2}, -ie^{-i\varphi} \cos \frac{\theta}{2}, \sin \frac{\theta}{2}\right), \\
A_{++}^\chi &= 2\left(-E_\nu e^{-i\varphi} \cos \frac{\theta}{2}, E_p \sin \frac{\theta}{2}, -iE_p \sin \frac{\theta}{2}, E_p e^{-i\varphi} \cos \frac{\theta}{2}\right) \\
&= (-)2\left(E_\nu e^{-i\varphi} \cos \frac{\theta}{2}, -E_p \sin \frac{\theta}{2}, iE_p \sin \frac{\theta}{2}, -E_p e^{-i\varphi} \cos \frac{\theta}{2}\right).
\end{aligned}$$

In [19], these components of the vector and axial nucleon currents in the helicity basis (indicated by the index χ) by considering the designations $c_{\theta/2} = \cos \frac{\theta}{2}$, $s_{\theta/2} = \sin \frac{\theta}{2}$ are written with different sign (see footnote 48)) as

$$\begin{aligned}
V_{++}^\chi &= 2\left(c_{\theta/2} e^{-i\varphi} E_p, (-s_{\theta/2}, is_{\theta/2}, -c_{\theta/2} e^{-i\varphi}) E_\nu\right), \quad V_{--}^\chi = 2\left(c_{\theta/2} e^{+i\varphi} E_p, (-s_{\theta/2}, -is_{\theta/2}, -c_{\theta/2} e^{+i\varphi}) E_\nu\right), \\
V_{+-}^\chi &= 2m\left(s_{\theta/2}, 0, 0, 0\right), \quad V_{-+}^\chi = 2m\left(-s_{\theta/2}, 0, 0, 0\right), \quad A_{++}^\chi = 2\left(c_{\theta/2} e^{-i\varphi} E_\nu, (-s_{\theta/2}, is_{\theta/2}, -c_{\theta/2} e^{-i\varphi}) E_p\right), \\
A_{--}^\chi &= 2\left(-c_{\theta/2} e^{+i\varphi} E_\nu, (s_{\theta/2}, is_{\theta/2}, c_{\theta/2} e^{+i\varphi}) E_p\right), \quad A_{+-}^\chi = 2m\left(0, -c_{\theta/2} e^{-\varphi}, -ic_{\theta/2} e^{-i\varphi}, s_{\theta/2}\right), \\
A_{-+}^\chi &= 2m\left(0, -c_{\theta/2} e^{+i\varphi}, ic_{\theta/2} e^{+i\varphi}, s_{\theta/2}\right).
\end{aligned} \tag{270}$$

³²The sign $(-)$ temporarily separated out below reflects the (abovementioned and insignificant) difference in the definition of the nucleon initial spinor $\chi_+(\mathbf{p}_i)$: $\chi_+(\mathbf{p}_i)_{\text{here}}$ in (246) = $-\chi_+(\mathbf{p}_i)$ in (83) from [19].

It is evident from these general expressions that $V_{+-} = V_{-+} = A_{+-} = A_{-+} = 0$ for massless neutrinos and antineutrinos, which means conservation of neutrino and antineutrino helicity in weak interactions. Therefore, e.g., in (264), only left-handed neutrino currents are needed for calculation of cross sections for elastic neutrino–nucleus scattering.

Thus, in the helicity basis the weak nucleon current at all possible helicities of the initial (second index) and final (first index) nucleons has the form

$$h_{\pm\pm}^{\mu} = \bar{u}(\mathbf{p}_f, \pm) \gamma^{\mu} (g_V - g_A \gamma_5) u(\mathbf{p}_i, \pm) = g_V \bar{u}(\mathbf{p}_f, \pm) \gamma^{\mu} u(\mathbf{p}_i, \pm) - g_A \bar{u}(\mathbf{p}_f, \pm) \gamma^{\mu} \gamma_5 u(\mathbf{p}_i, \pm) = g_V V_{\pm\pm}^{\chi, \mu} - g_A A_{\pm\pm}^{\chi, \mu}. \quad (271)$$

This nucleon current $h_{rr}^{\chi, \mu} = g_V V_{rr}^{\chi, \mu} - g_A A_{rr}^{\chi, \mu}$ can be written on a component-by-component basis as

$$\begin{aligned} h_{--}^{\chi, \mu=0,1,2,3}(\mathbf{p}', \mathbf{p}) &= 2 \left\{ e^{i\phi} (E_V g_A + E_p g_V) \cos \frac{\theta}{2}, - (E_p g_A + E_V g_V) \sin \frac{\theta}{2}, \right. \\ &\quad \left. -i(E_p g_A + E_V g_V) \sin \frac{\theta}{2}, -e^{i\phi} (E_p g_A + E_V g_V) \cos \frac{\theta}{2} \right\}, \\ h_{-+}^{\chi, \mu=0,1,2,3}(\mathbf{p}', \mathbf{p}) &= 2 \left\{ g_V m \sin \frac{\theta}{2}, -e^{i\phi} g_A m \cos \frac{\theta}{2}, i e^{i\phi} g_A m \cos \frac{\theta}{2}, g_A m \sin \frac{\theta}{2} \right\}, \\ h_{+-}^{\chi, \mu=0,1,2,3}(\mathbf{p}', \mathbf{p}) &= 2 \left\{ g_V m \sin \frac{\theta}{2}, e^{-i\phi} g_A m \cos \frac{\theta}{2}, i e^{-i\phi} g_A m \cos \frac{\theta}{2}, -g_A m \sin \frac{\theta}{2} \right\}, \\ h_{++}^{\chi, \mu=0,1,2,3}(\mathbf{p}', \mathbf{p}) &= 2 \left\{ e^{-i\phi} (E_V g_A - E_p g_V) \cos \frac{\theta}{2}, (-E_p g_A + E_V g_V) \sin \frac{\theta}{2}, \right. \\ &\quad \left. i(E_p g_A - E_V g_V) \sin \frac{\theta}{2}, e^{-i\phi} (-E_p g_A + E_V g_V) \cos \frac{\theta}{2} \right\}. \end{aligned} \quad (272)$$

It is these expressions that are necessary for calculating scalar products ($l_{\nu/\bar{\nu}}, h^{\chi}$).

10.4.2. Helicity basis: Scalar products of currents ($\mathbf{l}, \mathbf{h}$). Remember that the neutrino and antineutrino weak currents in the helicity basis have, according to (266) and (268), the following component-by-component form:

$$\begin{aligned} l_{\nu}^{\chi, \mu=0,1,2,3}(\mathbf{k}', \mathbf{k}) \\ = 4E_{\nu} \left(\cos \frac{\theta}{2}, e^{i\phi} \sin \frac{\theta}{2}, -i e^{i\phi} \sin \frac{\theta}{2}, \cos \frac{\theta}{2} \right), \end{aligned} \quad (273)$$

$$\begin{aligned} l_{\bar{\nu}}^{\chi, \mu=0,1,2,3}(\mathbf{k}', \mathbf{k}) \\ = 4E_{\nu} \left(\cos \frac{\theta}{2}, e^{-i\phi} \sin \frac{\theta}{2}, i e^{-i\phi} \sin \frac{\theta}{2}, \cos \frac{\theta}{2} \right). \end{aligned} \quad (274)$$

With formulas (272), (273), and (274) and the metric tensor $g_{\mu\nu}$ ³³, the desired scalar products of the leptonic and nucleon currents are calculated as follows:

$$\begin{aligned} (l_{\nu} \cdot h_{rr}^{\chi}) &\equiv \sum_{\mu, \nu} l_{\nu}^{\mu} g_{\mu\nu} h_{rr}^{\chi, \nu}, \\ (l_{\bar{\nu}} \cdot h_{rr}^{\chi}) &\equiv \sum_{\mu, \nu} l_{\bar{\nu}}^{\mu} g_{\mu\nu} h_{rr}^{\chi, \nu}. \end{aligned} \quad (275)$$

³³ $g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$. For neutrinos, $(l_{\nu}, h_{rr}^{\chi}) \bar{u}(k', -1) \gamma^{\mu} (1 - \gamma^5) u(k, -1) \bar{u}(p', r') \gamma_{\mu} (g_V - g_A \gamma^5) u(p, r)$.

These calculations can be performed directly, for example, as follows:

$$\begin{aligned} (l_{\nu} \cdot h_{rr}^{\chi}) &= \sum_{\mu, \nu} l_{\nu}^{\mu} g_{\mu\nu} h_{rr}^{\chi, \nu} 4E_{\nu} \left(\cos \frac{\theta}{2}, e^{i\phi} \sin \frac{\theta}{2}, \right. \\ &\quad \left. -i e^{i\phi} \sin \frac{\theta}{2}, \cos \frac{\theta}{2} \right) (g_{\mu\nu}) \\ &\times 2 \left\{ e^{i\phi} (E_V g_A + E_p g_V) \cos \frac{\theta}{2}, - (E_p g_A + E_V g_V) \sin \frac{\theta}{2}, \right. \\ &\quad \left. -i(E_p g_A + E_V g_V) \sin \frac{\theta}{2}, -e^{i\phi} (E_p g_A + E_V g_V) \cos \frac{\theta}{2} \right\} \\ &= 8E_{\nu} \left\{ \left[\cos \frac{\theta}{2} \right] \left[e^{i\phi} (E_V g_A + E_p g_V) \cos \frac{\theta}{2} \right] \right. \\ &\quad \left. - \left[e^{i\phi} \sin \frac{\theta}{2} \right] \left[- (E_p g_A + E_V g_V) \sin \frac{\theta}{2} \right] \right. \\ &\quad \left. - \left[-i e^{i\phi} \sin \frac{\theta}{2} \right] \left[-i (E_p g_A + E_V g_A) \sin \frac{\theta}{2} \right] \right. \\ &\quad \left. - \left[\cos \frac{\theta}{2} \right] \left[-e^{i\phi} (E_p g_A + E_V g_V) \cos \frac{\theta}{2} \right] \right\} \\ &\times 8E_{\nu} e^{i\phi} \left\{ (E_V g_A + E_p g_V) \cos^2 \frac{\theta}{2} \right. \\ &\quad \left. + (E_p g_A + E_V g_V) \sin^2 \frac{\theta}{2} \right. \\ &\quad \left. + (E_p g_A + E_V g_V) \sin^2 \frac{\theta}{2} + (E_p g_A + E_V g_V) \cos^2 \frac{\theta}{2} \right\} \\ &= 8E_{\nu} e^{i\phi} \left\{ (E_V g_A + E_p g_V) \left[1 - \sin^2 \frac{\theta}{2} \right] \right. \\ &\quad \left. + (E_p g_A + E_V g_V) \sin^2 \frac{\theta}{2} + (E_p g_A + E_V g_V) \right\} \\ &= O_{\nu} e^{i\phi} \left\{ E_V g_A + E_p g_V + E_p g_A + E_V g_V \right. \\ &\quad \left. + [E_p g_A + E_V g_V - E_V g_A - E_p g_V] \sin^2 \frac{\theta}{2} \right\} \end{aligned} \quad (277)$$

$$\begin{aligned}
&= 8E_\nu e^{i\phi} \left\{ [E_\nu + E_p][g_A + g_V] \right. \\
&\quad \left. + [E_p - E_\nu][g_A - g_V] \sin^2 \frac{\theta}{2} \right\} \\
&= 8 \frac{s-m^2}{2\sqrt{s}} e^{i\phi} \left\{ [\sqrt{s}][2g_L] - \left[\frac{m^2}{\sqrt{s}} \right] [2g_R] \sin^2 \frac{\theta}{2} \right\} \\
&\quad \times O^{i\phi}(s-m^2) \left(g_L - g_R \frac{m^2}{s} \sin^2 \frac{\theta}{2} \right) \rightarrow
\end{aligned}$$

As a result, in the helicity basis (and in the lepton–nucleon center-of-mass system), by considering

$$g_{V/A} = g_L \pm g_R, \quad E_{p/\nu} = \frac{s \pm m^2}{2\sqrt{s}} \quad (276)$$

$$\text{and } \cos \theta = 1 - 2 \sin^2 \frac{\theta}{2},$$

the scalar products of the neutrino and nucleon currents for all possible helicities of the initial and final nucleons have the form

$$\begin{aligned}
(l_\nu \cdot h_{++}^\chi) &= (-) 8e^{-i\phi} g_R (s-m^2) \left(1 - \sin^2 \frac{\theta}{2} \right), \\
(l_\nu \cdot h_{--}^\chi) &= 8e^{i\phi} (s-m^2) \left(g_L - g_R \sin^2 \frac{\theta}{2} \frac{m^2}{s} \right), \\
(l_\nu \cdot h_{+-}^\chi) &= 8g_R (s-m^2) \frac{m}{\sqrt{s}} \sin \frac{\theta}{2} \cos \frac{\theta}{2}, \\
(l_\nu \cdot h_{-+}^\chi) &= (+) 8g_R (s-m^2) \frac{m}{\sqrt{s}} \sin \frac{\theta}{2} \cos \frac{\theta}{2}.
\end{aligned} \quad (277)$$

In [19], expressions for (l, h_{++}^χ) and (l, h_{-+}^χ) are given with different sign because, as was already mentioned, $\chi_+(\mathbf{p}_i)_{\text{here}} = -\chi_+(\mathbf{p}_i)$ in [19]

$$\begin{aligned}
(l, h_{++}^\chi) &= 8(s-m^2) e^{-i\phi} \cos^2 \frac{\theta}{2} g_R, \\
(l, h_{--}^\chi) &= 8(s-m^2) e^{+i\phi} \left(g_L - g_R \sin^2 \frac{\theta}{2} \frac{m^2}{s} \right), \\
(l, h_{+-}^\chi) &= 8(s-m^2) \frac{m}{\sqrt{s}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} g_R, \\
(l, h_{-+}^\chi) &= -8(s-m^2) \frac{m}{\sqrt{s}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} g_R.
\end{aligned} \quad (278)$$

This difference in sign disappears in squares of scalar products that enter into the observables.

Squares of scalar products $|(l_\nu \cdot h_{r,r'}^\chi)|^2$ and their total sum over the initial and final nucleon spin projections (all helicities) $\sum_{r,r'} |(l_\nu \cdot h_{r,r'}^\chi)|^2$ are given by the expressions

$$\begin{aligned}
|(l_\nu \cdot h_{--}^\chi)|^2 &= \frac{16}{s^2} (m^2 - s)^2 \\
&\quad \times (-g_R m^2 + 2g_L s + g_R m^2 \cos \theta)^2, \\
|(l_\nu \cdot h_{+-}^\chi)|^2 &= \frac{16}{s} g_R^2 m^2 (m^2 - s)^2 \sin^2 \theta, \\
|(l_\nu \cdot h_{-+}^\chi)|^2 &= \frac{16}{s} g_R^2 m^2 (m^2 - s)^2 \sin^2 \theta,
\end{aligned}$$

$$\begin{aligned}
|(l_\nu \cdot h_{++}^\chi)|^2 &= 16g_R^2 (m^2 - s)^2 (1 + \cos \theta)^2; \\
\sum_{r,r'} |(l_\nu \cdot h_{r,r'}^\chi)|^2 &= g_L^2 + \frac{g_L g_R m^2 (-1 + \cos \theta)}{s} \\
&\quad + \frac{g_R^2 (m^2 + s + (-m^2 + s) \cos \theta)^2}{4s^2} \\
&= g_L^2 + g_R^2 (1 - y)^2 - 2g_L g_R \frac{ym^2}{s - m^2}.
\end{aligned}$$

The last line corresponds to the known formula for neutrino–nucleon scattering with Z^0 boson exchange (see, for example, [42, 163])

$$\begin{aligned}
\sum_{r,r'} |(l, h_{r,r'}^\chi)|^2 &= 2^6 (s - m^2)^2 \\
&\times \left(g_L^2 + g_R^2 (1 - y)^2 - 2g_L g_R \frac{ym^2}{s - m^2} \right).
\end{aligned} \quad (279)$$

In the neutrino and nucleon center-of-mass system, the Bjorken variable y and $\sin^2 \frac{\theta}{2}$ are connected by the relations

$$\begin{aligned}
y &= \frac{(p, q)}{(p, k)}, \quad \sin^2 \frac{\theta}{2} = \frac{ys}{s - m^2}, \\
\cos \theta &= 1 - \frac{2ys}{s - m^2}.
\end{aligned} \quad (280)$$

Similar to (277), by considering (274) and substitutions (276), scalar products of the antineutrino and nucleon currents in the helicity basis for all possible helicities of the initial and final nucleons are

$$\begin{aligned}
(l_{\bar{\nu}} \cdot h_{--}^\chi) &= 4e^{i\phi} g_L (-m^2 + s) (1 + \cos \theta), \\
(l_{\bar{\nu}} \cdot h_{+-}^\chi) &= \frac{4g_L m (-m^2 + s) \sin \theta}{\sqrt{s}}, \\
(l_{\bar{\nu}} \cdot h_{-+}^\chi) &= \frac{4g_L m (-m^2 + s) \sin \theta}{\sqrt{s}}, \\
(l_{\bar{\nu}} \cdot h_{++}^\chi) &= \frac{4e^{-i\phi} (m^2 - s) (-g_L m^2 + 2g_R s + g_L m^2 \cos \theta)}{s}.
\end{aligned} \quad (281)$$

Squares of these (antineutrino) scalar products $|(l_{\bar{\nu}} \cdot h_{r,r'}^\chi)|^2$ and their total sum over the initial and final nucleon spin projections (all nucleon helicities) $\sum_{r,r'} |(l_{\bar{\nu}} \cdot h_{r,r'}^\chi)|^2$ take the form

$$\begin{aligned}
|(l_{\bar{\nu}} \cdot h_{--}^\chi)|^2 &= 16g_L^2 (m^2 - s)^2 (1 + \cos \theta)^2, \\
|(l_{\bar{\nu}} \cdot h_{+-}^\chi)|^2 &= 16g_L^2 \frac{m^2}{s} (m^2 - s)^2 \sin^2 \theta, \\
|(l_{\bar{\nu}} \cdot h_{-+}^\chi)|^2 &= 16g_L^2 \frac{m^2}{s} (m^2 - s)^2 \sin^2 \theta, \\
|(l_{\bar{\nu}} \cdot h_{++}^\chi)|^2 &= \frac{16(m^2 - s)^2 (-g_L m^2 + 2g_R s + g_L m^2 \cos \theta)^2}{s^2};
\end{aligned} \quad (282)$$

$$\begin{aligned} \frac{\sum_{r,r'} |(l_{\bar{\nu}} \cdot h_{r,r}^x)|^2}{64(m^2 - s)^2} &= g_R^2 + \frac{g_L g_R m^2 (-1 + \cos \theta)}{s} \\ &+ \frac{g_L^2 (m^2 + s + (-m^2 + s) \cos \theta)^2}{4s^2} \\ &= g_R^2 + g_L^2 (1 - y)^2 - 2g_L g_R \frac{ym^2}{s - m^2}. \end{aligned}$$

Expression (282) corresponds to the known formula for the antineutrino scattering off the nucleon (electron) that can be derived from the expression for the neutrino–nucleon (electron) scattering (279) by substituting g_R for g_L and vice versa (see, for example, [42, 163]).

Thus, formulas (277) and (281) are the desired set of the scalar products of the lepton–nucleon weak currents in the center-of-mass system of colliding particles, which is obtained in the helicity basis for all possible helicities of the initial (second index $\pm$) and final (first index $\pm$) nucleons. Formulas (277) correspond to the neutrino–nucleon scattering, and formulas (281) correspond to the antineutrino–nucleon scattering.

10.4.3. σ_3 Basis: Eigenvectors and components of currents. As above, we assume that the direction of the incident (on a nucleus and/or a nucleon) initial (anti)neutrino coincides with the z axis (Fig. 21) and is specified by the unit vector $\mathbf{n}_{k_i} = (0, 0, 1)$. The helicity operator for the initial neutrino and antineutrino

$\alpha_{k_i} = \mathbf{n}_{k_i} \cdot \boldsymbol{\sigma} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ has eigenvalues -1 and $+1$,

and corresponding normalized, $|\eta_{\pm}(\mathbf{k}_i)|^2 = 1$, eigenvectors are

$$\eta_{-}(\mathbf{k}_i) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \eta_{+}(\mathbf{k}_i) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (283)$$

As previously, the direction of motion of the final (anti)neutrino is specified by the unit vector $\mathbf{n}_{k_f} = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta)$. The helicity operator for the final lepton has the standard form

$$\alpha_{k_f} = \mathbf{n}_{k_f} \cdot \boldsymbol{\sigma} = \begin{pmatrix} \cos \theta & e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & -\cos \theta \end{pmatrix}.$$

To its eigenvalues -1 and $+1$ their correspond normalized, $|\eta_{\pm}(\mathbf{k}_f)|^2 = 1$, eigenvectors

$$\eta_{-}(\mathbf{k}_f) \equiv \chi_{-}(\mathbf{k}_f) \quad \text{and} \quad \eta_{+}(\mathbf{k}_f) \equiv \chi_{+}(\mathbf{k}_f), \quad (284)$$

the explicit form of which is defined by expressions (246). In other words, since the neutrino and antineutrino helicity is conserved, the eigenstates in the helicity basis (of the initial neutral massless lepton) and the σ_3 basis coincide.

The situation is different for a massive nucleon. Now we will quantize the nucleon spin not along the

direction of motion of the nucleon itself (helicity basis) but along the $\mathbf{n}_k$ axis of the incident (anti)neutrino in the lepton–nucleon center-of-mass system. This means that bispinors of the initial $\eta_r(\mathbf{p}_i)$ and final $\eta_{r'}(\mathbf{p}_f)$ nucleons are eigenvectors of the operator $\mathbf{n}_k \times \boldsymbol{\sigma} = \sigma_3$ (hence the σ_3 basis)

$$\begin{aligned} \sigma_3 \eta_r(\mathbf{p}_i) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \eta_r(\mathbf{p}_i) = r \eta_r(\mathbf{p}_i), \\ \sigma_3 \eta_{r'}(\mathbf{p}_f) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \eta_{r'}(\mathbf{p}_f) = r' \eta_{r'}(\mathbf{p}_f). \end{aligned}$$

As a result, for the initial nucleon the eigenvectors $\eta_{\mp}(\mathbf{p}_i)$ correspond to the eigenvalues $r = \mp 1$. Since the spin of both the initial and final nucleon is quantized along the same axis (i.e., it retains its projection on this axis or changes the projection to the strictly opposite one) specified by the direction of the incident lepton $\mathbf{n}_{k_i}$, for the eigenvalues of the final nucleon spin projection $r' = \mp 1$ there are eigenvectors $\eta_{\mp}(\mathbf{p}_f) = \eta_{\mp}(\mathbf{p}_i)$ normalized by the condition $|\eta_{\pm}(\mathbf{p}_{f,i})|^2 = 1$, or, in the explicit form,

$$\begin{aligned} \eta_{+} &= \eta_{+}(\mathbf{p}_i) = \eta_{+}(\mathbf{p}_f) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ \eta_{-} &= \eta_{-}(\mathbf{p}_i) = \eta_{-}(\mathbf{p}_f) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned} \quad (285)$$

Thus, for calculating all vector and axial-vector (anti)neutrino and nucleon components of currents in the σ_3 basis, when the nucleon spin is quantized along the incident (anti)neutrino axis, the following basis bispinors normalized by the condition $|\eta|^2 = 1$ are necessary:

$$\begin{aligned} \text{Initial: } \eta_{-}(\mathbf{k}_i) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv \chi_{-}(\mathbf{k}_i), \\ \eta_{+}(\mathbf{k}_i) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv \chi_{+}(\mathbf{k}_i); \quad \eta_{-}(\mathbf{p}_i) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv \chi_{-}(\mathbf{p}_i), \\ \eta_{+}(\mathbf{p}_i) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv \chi_{+}(\mathbf{p}_i), \\ \text{Final: } \eta_{-}(\mathbf{p}_f) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv \eta_{-}(\mathbf{p}_i), \\ \eta_{+}(\mathbf{p}_f) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv \eta_{+}(\mathbf{p}_i), \\ \eta_{-}(\mathbf{k}_f) &= \cos \frac{\theta}{2} \begin{pmatrix} -e^{-i\phi} \tan \frac{\theta}{2} \\ 1 \end{pmatrix} \equiv \chi_{-}(\mathbf{k}_f), \\ \eta_{+}(\mathbf{k}_f) &= e^{i\phi} \sin \frac{\theta}{2} \begin{pmatrix} e^{-i\phi} \cot \frac{\theta}{2} \\ 1 \end{pmatrix} \equiv \chi_{+}(\mathbf{k}_f). \end{aligned} \quad (286)$$

It is evident from these formulas that basis spinors for initial and final neutral (massless) leptons are identical in both bases. It is different for massive nucleons. Since in the lepton–nucleon center-of-mass system $\mathbf{k}_i = -\mathbf{p}_i$, spinors $\eta_{\mp}$ of initial nucleons in the σ_3 basis (projection on direction $\mathbf{k}_i$) correspond to opposite eigenvalues of spinors $\chi_{\pm}$ from the helicity basis (projection on direction $\mathbf{p}_i$).

σ_3 Basis: Components of leptonic currents

According to the consideration in the helicity basis, the current matrices $\hat{V}^{0,1,2,3}(\mathbf{k}_f, \mathbf{k}_i)$ and $\hat{A}^{0,1,2,3}(\mathbf{k}_f, \mathbf{k}_i)$ ³⁴ are identical for the neutrino and the antineutrino and have the form (262) and (263)

$$\begin{aligned}\hat{V}^0(\mathbf{k}_f, \mathbf{k}_i) &= E_v \begin{pmatrix} 1 + \cos \theta & -e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & 1 + \cos \theta \end{pmatrix}, \\ \hat{V}^1(\mathbf{k}_f, \mathbf{k}_i) &= E_v \begin{pmatrix} e^{-i\phi} \sin \theta & -1 + \cos \theta \\ 1 - \cos \theta & e^{i\phi} \sin \theta \end{pmatrix}, \\ \hat{V}^2(\mathbf{k}_f, \mathbf{k}_i) &= E_v \begin{pmatrix} ie^{-i\phi} \sin \theta & i - i \cos \theta \\ i - i \cos \theta & -ie^{i\phi} \sin \theta \end{pmatrix}, \\ \hat{V}^3(\mathbf{k}_f, \mathbf{k}_i) &= E_v \begin{pmatrix} 1 + \cos \theta & -e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & 1 + \cos \theta \end{pmatrix}; \\ \hat{A}^0(\mathbf{k}_f, \mathbf{k}_i) &= E_v \begin{pmatrix} 1 + \cos \theta & e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & -1 - \cos \theta \end{pmatrix}, \\ \hat{A}^1(\mathbf{k}_f, \mathbf{k}_i) &= E_v \begin{pmatrix} e^{-i\phi} \sin \theta & 1 - \cos \theta \\ 1 - \cos \theta & -e^{i\phi} \sin \theta \end{pmatrix}, \\ \hat{A}^2(\mathbf{k}_f, \mathbf{k}_i) &= E_v \begin{pmatrix} ie^{-i\phi} \sin \theta & -i(1 - \cos \theta) \\ i(1 - \cos \theta) & ie^{i\phi} \sin \theta \end{pmatrix}, \\ \hat{A}^3(\mathbf{k}_f, \mathbf{k}_i) &= E_v \begin{pmatrix} 1 + \cos \theta & e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & -1 - \cos \theta \end{pmatrix}.\end{aligned}$$

Using these matrices for the (left-handed) neutrino with $\eta_{-}(\mathbf{k}_{i/f})$ and (right-handed) antineutrino with $\eta_{+}(\mathbf{k}_{i/f})$ in the σ_3 basis, one should calculate components of the vector and axial-vector currents

$$\begin{aligned}V_{--}^{\mu}(\mathbf{k}_f, \mathbf{k}_i) &= \eta_{-}^{\dagger}(\mathbf{k}_f) \hat{V}^{\mu}(\mathbf{k}_f, \mathbf{k}_i) \eta_{-}(\mathbf{k}_i), \\ A_{--}^{\mu}(\mathbf{k}_f, \mathbf{k}_i) &= \eta_{-}^{\dagger}(\mathbf{k}_f) \hat{A}^{\mu}(\mathbf{k}_f, \mathbf{k}_i) \eta_{-}(\mathbf{k}_i), \\ V_{++}^{\mu}(\mathbf{k}_f, \mathbf{k}_i) &= \eta_{+}^{\dagger}(\mathbf{k}_f) \hat{V}^{\mu}(\mathbf{k}_f, \mathbf{k}_i) \eta_{+}(\mathbf{k}_i), \\ A_{++}^{\mu}(\mathbf{k}_f, \mathbf{k}_i) &= \eta_{+}^{\dagger}(\mathbf{k}_f) \hat{A}^{\mu}(\mathbf{k}_f, \mathbf{k}_i) \eta_{+}(\mathbf{k}_i).\end{aligned}$$

Since, according to (284), the explicit form of $\eta_{\pm}(\mathbf{k}_{i/f})$ coincides with $\chi_{\pm}(\mathbf{k}_{i/f})$, expressions for leptonic currents in the σ_3 basis should also coincide with

the earlier obtained expressions for these currents in the helicity basis, and consequently, the total neutrino and antineutrino weak currents have the same form

$$l_v(\mathbf{k}_f, \mathbf{k}_i) = 4E_v \left(\cos \frac{\theta}{2}, e^{i\phi} \sin \frac{\theta}{2}, -ie^{i\phi} \sin \frac{\theta}{2}, \cos \frac{\theta}{2} \right); \quad (287)$$

$$l_{\bar{v}}(\mathbf{k}_f, \mathbf{k}_i) = 4E_v \left(\cos \frac{\theta}{2}, e^{-i\phi} \sin \frac{\theta}{2}, ie^{-i\phi} \sin \frac{\theta}{2}, \cos \frac{\theta}{2} \right). \quad (288)$$

By virtue of the strict conservation of the massless neutrino and antineutrino helicity, the use of the σ_3 basis is significant only for nucleon currents.

σ_3 Basis: Components of nucleon currents

In the σ_3 basis, the Dirac four-spinors of the initial (with $\mathbf{p} \equiv \mathbf{p}_i$) and final (with $\mathbf{p}' \equiv \mathbf{p}_f$) nucleons needed for calculation of weak currents by formulas (251) have the form

$$\begin{aligned}u(\mathbf{p}, r) &= \begin{pmatrix} \lambda_{+} \\ \lambda_{-} \alpha_p \end{pmatrix} \eta_r(\mathbf{p}), \\ u^{\dagger}(\mathbf{p}', r') &= \eta_r^{\dagger}(\mathbf{p}') (\lambda_{+}, \lambda_{-} \alpha_{p'}^{\dagger}).\end{aligned} \quad (289)$$

According to kinematics of elastic scattering in the lepton–nucleon center-of-mass system, the nucleon helicity operators α_p and $\alpha_{p'}$, entering into expressions (289) coincide with the nucleon helicity operators in the helicity basis and are expressed as

$$\begin{aligned}\alpha_p &= \alpha_{p_i} = \mathbf{n}_{p_i} \cdot \boldsymbol{\sigma} = -\sigma_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \\ \alpha_{p'} &= \alpha_{p_f} = \mathbf{n}_{p_f} \cdot \boldsymbol{\sigma} = \begin{pmatrix} -\cos \theta & -e^{-i\phi} \sin \theta \\ -e^{i\phi} \sin \theta & \cos \theta \end{pmatrix}.\end{aligned}$$

Then, from the definition of currents (251) we have in the σ_3 basis

$$\begin{aligned}V_{r'r}^{\mu}(\mathbf{p}', \mathbf{p}) &= \bar{u}(\mathbf{p}', r') \gamma^{\mu} u(\mathbf{p}, r) \\ &= \eta_r^{\dagger}(\mathbf{p}') \hat{V}^{\mu}(\mathbf{p}_f, \mathbf{p}_i) \eta_r(\mathbf{p}) \equiv \eta_r^{\dagger} \hat{V}^{\mu}(\mathbf{p}_f) \eta_r, \\ A_{r'r}^{\mu}(\mathbf{p}', \mathbf{p}) &= \bar{u}(\mathbf{p}', r') \gamma^{\mu} \gamma_5 u(\mathbf{p}, r) \\ &= \eta_r^{\dagger}(\mathbf{p}') \hat{A}^{\mu}(\mathbf{p}_f, \mathbf{p}_i) \eta_r(\mathbf{p}) \equiv \eta_r^{\dagger} \hat{A}^{\mu}(\mathbf{p}_f) \eta_r,\end{aligned} \quad (290)$$

where matrices of currents are expressed in terms of matrices of initial and final nucleon helicity

$$\begin{aligned}\hat{V}^0(\mathbf{p}_f) &\equiv \lambda_{+}^2 + \lambda_{-}^2 \alpha_p \alpha_{p'}, \\ \hat{\mathbf{V}}(\mathbf{p}_f) &\equiv \lambda_{-} \lambda_{+} [\alpha_p \boldsymbol{\sigma} + \sigma \alpha_{p'}], \\ \hat{A}^0(\mathbf{p}_f) &\equiv \lambda_{+} \lambda_{-} [\alpha_p \boldsymbol{\sigma} + \sigma \alpha_{p'}], \\ \hat{\mathbf{A}}(\mathbf{p}_f) &\equiv \lambda_{+}^2 \sigma + \lambda_{-}^2 \alpha_p \sigma \alpha_{p'}.\end{aligned}$$

Since in the center-of-mass system $\lambda_{\pm}^2 = E_p \pm m$ and $\lambda_{+} \lambda_{-} = \sqrt{E_p^2 - m^2} = E_v$, and by considering the explicit form of $\alpha_{p'}$ and α_p , in the σ_3 basis we have expressions for the matrices $\hat{V}^{\mu}(\mathbf{p}_f)$ and $\hat{A}^{\mu}(\mathbf{p}_f)$ completely coinciding with those earlier obtained in the helicity basis

³⁴By virtue of the explicit equations for the four-spinors $u(\mathbf{k})$ and $v(\mathbf{k})$, and procedures for deriving formulas (262) and (263).

$$\begin{aligned}
\hat{V}^0(\mathbf{p}_f) &= \begin{pmatrix} E_p + m + (E_p - m) \cos \theta & -e^{-i\phi}(E_p - m) \sin \theta \\ e^{i\phi}(E_p - m) \sin \theta & E_p + m + (E_p - m) \cos \theta \end{pmatrix}, \\
\hat{V}^1(\mathbf{p}_f) &= E_v \begin{pmatrix} -e^{-i\phi} \sin \theta & 1 - \cos \theta \\ -1 + \cos \theta & -e^{i\phi} \sin \theta \end{pmatrix}, \\
\hat{V}^2(\mathbf{p}_f) &= iE_v \begin{pmatrix} -e^{-i\phi} \sin \theta & -1 + \cos \theta \\ -1 + \cos \theta & e^{i\phi} \sin \theta \end{pmatrix}, \\
\hat{V}^3(\mathbf{p}_f) &= E_v \begin{pmatrix} -(1 + \cos \theta) & e^{-i\phi} \sin \theta \\ -e^{i\phi} \sin \theta & -(1 + \cos \theta) \end{pmatrix}; \\
\hat{A}^0(\mathbf{p}_f) &= E_v \begin{pmatrix} -(1 + \cos \theta) & -e^{-i\phi} \sin \theta \\ -e^{i\phi} \sin \theta & (1 + \cos \theta) \end{pmatrix}, \\
\hat{A}^1(\mathbf{p}_f) &= \begin{pmatrix} (E_p - m)e^{-i\phi} \sin \theta & (E_p + m) - (E_p - m) \cos \theta \\ (E_p + m) - (E_p - m) \cos \theta & -(E_p - m)e^{i\phi} \sin \theta \end{pmatrix}, \\
\hat{A}^2(\mathbf{p}_f) &= i \begin{pmatrix} (E_p - m)e^{-i\phi} \sin \theta & -(E_p + m) + (E_p - m) \cos \theta \\ (E_p + m) - (E_p - m) \cos \theta & (E_p - m)e^{i\phi} \sin \theta \end{pmatrix}, \\
\hat{A}^3(\mathbf{p}_f) &= \begin{pmatrix} (E_p + m) + (E_p - m) \cos \theta & (E_p - m)e^{-i\phi} \sin \theta \\ (E_p - m)e^{i\phi} \sin \theta & -(E_p + m) - (E_p - m) \cos \theta \end{pmatrix}.
\end{aligned}$$

Substituting these expressions into the right-hand sides of the definitions of currents in the σ_3 basis (290) and considering

$$\begin{aligned}
\eta_- &\equiv \eta_-(\mathbf{p}_i) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \eta_+ \equiv \eta_+(\mathbf{p}_i) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\
\eta_-^\dagger &\equiv \eta_-^\dagger(\mathbf{p}_f) = (0 \ 1), \quad \eta_+^\dagger \equiv \eta_+^\dagger(\mathbf{p}_f) = (1 \ 0),
\end{aligned}$$

we obtain explicit expressions for all components of the nucleon current at all possible nucleon spin projections on the direction of motion of the initial lepton

$$\begin{aligned}
V_{--}^0(\mathbf{p}_f) &= \eta_-^\dagger \hat{V}^0(\mathbf{p}_f) \eta_- = (01) \begin{pmatrix} E_p + m + (E_p - m) \cos \theta & -e^{-i\phi}(E_p - m) \sin \theta \\ e^{i\phi}(E_p - m) \sin \theta & E_p + m + (E_p - m) \cos \theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= (01) \begin{pmatrix} -e^{-i\phi}(E_p - m) \sin \theta \\ E_p + m + (E_p - m) \cos \theta \end{pmatrix} = E_p + m + (E_p - m) \cos \theta, \\
V_{--}^1(\mathbf{p}_f) &= \eta_-^\dagger \hat{V}^1(\mathbf{p}_f) \eta_- = (01) E_v \begin{pmatrix} -e^{-i\phi} \sin \theta & 1 - \cos \theta \\ -1 + \cos \theta & -e^{i\phi} \sin \theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -E_v e^{i\phi} \sin \theta, \\
V_{--}^2(\mathbf{p}_f) &= \eta_-^\dagger \hat{V}^2(\mathbf{p}_f) \eta_- = (01) iE_v \begin{pmatrix} -e^{-i\phi} \sin \theta & -1 + \cos \theta \\ -1 + \cos \theta & e^{i\phi} \sin \theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = iE_v e^{i\phi} \sin \theta, \\
V_{--}^3(\mathbf{p}_f) &= \eta_-^\dagger \hat{V}^3(\mathbf{p}_f) \eta_- = (01) E_v \begin{pmatrix} -(1 + \cos \theta) & e^{-i\phi} \sin \theta \\ -e^{i\phi} \sin \theta & -(1 + \cos \theta) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -E_v (1 + \cos \theta); \\
A_{--}^0(\mathbf{p}_f) &= \eta_-^\dagger \hat{A}^0(\mathbf{p}_f) \eta_- = (01) E_v \begin{pmatrix} -(1 + \cos \theta) & -e^{-i\phi} \sin \theta \\ -e^{i\phi} \sin \theta & (1 + \cos \theta) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = E_v (1 + \cos \theta), \\
A_{--}^1(\mathbf{p}_f) &= \eta_-^\dagger \hat{A}^1(\mathbf{p}_f) \eta_- = (01) \begin{pmatrix} (E_p - m)e^{-i\phi} \sin \theta & (E_p + m) - (E_p - m) \cos \theta \\ (E_p + m) - (E_p - m) \cos \theta & -(E_p - m)e^{i\phi} \sin \theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= -(E_p - m)e^{i\phi} \sin \theta,
\end{aligned}$$

$$\begin{aligned}
A_{-}^2(\mathbf{p}_f) &= \eta_{-}^{\dagger} \hat{A}^2(\mathbf{p}_f) \eta_{-} = (01)i \begin{pmatrix} (E_p - m)e^{-i\phi} \sin \theta & -(E_p + m) + (E_p - m) \cos \theta \\ (E_p + m) - (E_p - m) \cos \theta & (E_p - m)e^{i\phi} \sin \theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= i(E_p - m)e^{i\phi} \sin \theta, \\
A_{-}^3(\mathbf{p}_f) &= \eta_{-}^{\dagger} \hat{A}^3(\mathbf{p}_f) \eta_{-} = (01) \begin{pmatrix} (E_p + m) + (E_p - m) \cos \theta & (E_p - m)e^{-i\phi} \sin \theta \\ (E_p - m)e^{i\phi} \sin \theta & -(E_p + m) - (E_p - m) \cos \theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= -(E_p + m) - (E_p - m) \cos \theta; \\
V_{-+}^0(\mathbf{p}_f) &= \eta_{-}^{\dagger} \hat{V}^0(\mathbf{p}_f) \eta_{+} = (01) \begin{pmatrix} E_p + m + (E_p - m) \cos \theta & -e^{-i\phi}(E_p - m) \sin \theta \\ e^{i\phi}(E_p - m) \sin \theta & E_p + m + (E_p - m) \cos \theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= (01) \begin{pmatrix} E_p + m + (E_p - m) \cos \theta \\ e^{i\phi}(E_p - m) \sin \theta \end{pmatrix} = e^{i\phi}(E_p - m) \sin \theta, \\
V_{-+}^1(\mathbf{p}_f) &= \eta_{-}^{\dagger} \hat{V}^1(\mathbf{p}_f) \eta_{+} = (01)E_v \begin{pmatrix} -e^{-i\phi} \sin \theta & 1 - \cos \theta \\ -1 + \cos \theta & -e^{i\phi} \sin \theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -E_v(1 - \cos \theta), \\
V_{-+}^2(\mathbf{p}_f) &= \eta_{-}^{\dagger} \hat{V}^2(\mathbf{p}_f) \eta_{+} = (01)iE_v \begin{pmatrix} -e^{-i\phi} \sin \theta & -1 + \cos \theta \\ -1 + \cos \theta & e^{i\phi} \sin \theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = iE_v(-1 + \cos \theta), \\
V_{-+}^3(\mathbf{p}_f) &= \eta_{-}^{\dagger} \hat{V}^3(\mathbf{p}_f) \eta_{+} = (01)E_v \begin{pmatrix} -(1 + \cos \theta) & e^{-i\phi} \sin \theta \\ -e^{i\phi} \sin \theta & -(1 + \cos \theta) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -e^{i\phi}E_v \sin \theta; \\
A_{+}^0(\mathbf{p}_f) &= \eta_{+}^{\dagger} \hat{A}^0(\mathbf{p}_f) \eta_{+} = (01)E_v \begin{pmatrix} -(1 + \cos \theta) & -e^{-i\phi} \sin \theta \\ -e^{i\phi} \sin \theta & (1 + \cos \theta) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -e^{i\phi}E_v \sin \theta, \\
A_{+}^1(\mathbf{p}_f) &= \eta_{+}^{\dagger} \hat{A}^1(\mathbf{p}_f) \eta_{+} = (01) \begin{pmatrix} (E_p - m)e^{-i\phi} \sin \theta & (E_p + m) - (E_p - m) \cos \theta \\ (E_p + m) - (E_p - m) \cos \theta & -(E_p - m)e^{i\phi} \sin \theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= (E_p + m) - (E_p - m) \cos \theta, \\
A_{+}^2(\mathbf{p}_f) &= \eta_{+}^{\dagger} \hat{A}^2(\mathbf{p}_f) \eta_{+} = (01)i \begin{pmatrix} (E_p - m)e^{-i\phi} \sin \theta & -(E_p + m) + (E_p - m) \cos \theta \\ (E_p + m) - (E_p - m) \cos \theta & (E_p - m)e^{i\phi} \sin \theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
A_{+}^3(\mathbf{p}_f) &= \eta_{+}^{\dagger} \hat{A}^3(\mathbf{p}_f) \eta_{+} = (01) \begin{pmatrix} (E_p + m) + (E_p - m) \cos \theta & (E_p - m)e^{-i\phi} \sin \theta \\ (E_p - m)e^{i\phi} \sin \theta & -(E_p + m) - (E_p - m) \cos \theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (E_p - m)e^{i\phi} \sin \theta; \\
V_{+-}^0(\mathbf{p}_f) &= \eta_{+}^{\dagger} \hat{V}^0(\mathbf{p}_f) \eta_{-} = (10) \begin{pmatrix} E_p + m + (E_p - m) \cos \theta & -e^{-i\phi}(E_p - m) \sin \theta \\ e^{i\phi}(E_p - m) \sin \theta & E_p + m + (E_p - m) \cos \theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= (10) \begin{pmatrix} -e^{-i\phi}(E_p - m) \sin \theta \\ E_p + m + (E_p - m) \cos \theta \end{pmatrix} = -e^{-i\phi}(E_p - m) \sin \theta, \\
V_{+-}^1(\mathbf{p}_f) &= \eta_{+}^{\dagger} \hat{V}^1(\mathbf{p}_f) \eta_{-} = (10)E_v \begin{pmatrix} -e^{-i\phi} \sin \theta & 1 - \cos \theta \\ -1 + \cos \theta & -e^{i\phi} \sin \theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = E_v(1 - \cos \theta), \\
V_{+-}^2(\mathbf{p}_f) &= \eta_{+}^{\dagger} \hat{V}^2(\mathbf{p}_f) \eta_{-} = (10)iE_v \begin{pmatrix} -e^{-i\phi} \sin \theta & -1 + \cos \theta \\ -1 + \cos \theta & e^{i\phi} \sin \theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = iE_v(-1 + \cos \theta), \\
V_{+-}^3(\mathbf{p}_f) &= \eta_{+}^{\dagger} \hat{V}^3(\mathbf{p}_f) \eta_{-} = (10)E_v \begin{pmatrix} -(1 + \cos \theta) & e^{-i\phi} \sin \theta \\ -e^{i\phi} \sin \theta & -(1 + \cos \theta) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = E_v e^{-i\phi} \sin \theta; \\
A_{+-}^0(\mathbf{p}_f) &= \eta_{+}^{\dagger} \hat{A}^0(\mathbf{p}_f) \eta_{-} = (10)E_v \begin{pmatrix} -(1 + \cos \theta) & -e^{-i\phi} \sin \theta \\ -e^{i\phi} \sin \theta & (1 + \cos \theta) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -e^{-i\phi}E_v \sin \theta,
\end{aligned}$$

$$\begin{aligned}
A_{+-}^1(\mathbf{p}_f) &= \eta_+^\dagger \hat{A}^1(\mathbf{p}_f) \eta_- = (10) \begin{pmatrix} (E_p - m)e^{-i\phi} \sin \theta & (E_p + m) - (E_p - m) \cos \theta \\ (E_p + m) - (E_p - m) \cos \theta & -(E_p - m)e^{i\phi} \sin \theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= (E_p + m) - (E_p - m) \cos \theta, \\
A_{+-}^2(\mathbf{p}_f) &= \eta_+^\dagger \hat{A}^2(\mathbf{p}_f) \eta_- = (10)i \begin{pmatrix} (E_p - m)e^{-i\phi} \sin \theta & -(E_p + m) + (E_p - m) \cos \theta \\ (E_p + m) - (E_p - m) \cos \theta & (E_p - m)e^{i\phi} \sin \theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= -i(E_p + m) + i(E_p - m) \cos \theta, \\
A_{+-}^3(\mathbf{p}_f) &= \eta_+^\dagger \hat{A}^3(\mathbf{p}_f) \eta_- = (10) \begin{pmatrix} (E_p + m) + (E_p - m) \cos \theta & (E_p - m)e^{-i\phi} \sin \theta \\ (E_p - m)e^{i\phi} \sin \theta & -(E_p + m) - (E_p - m) \cos \theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= (E_p - m)e^{-i\phi} \sin \theta; \\
V_{++}^0(\mathbf{p}_f) &= \eta_+^\dagger \hat{V}^0(\mathbf{p}_f) \eta_+ = (10) \begin{pmatrix} E_p + m + (E_p - m) \cos \theta & -e^{-i\phi} (E_p - m) \sin \theta \\ e^{i\phi} (E_p - m) \sin \theta & E_p + m + (E_p - m) \cos \theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= (10) \begin{pmatrix} E_p + m + (E_p - m) \cos \theta \\ e^{i\phi} (E_p - m) \sin \theta \end{pmatrix} = E_p + m + (E_p - m) \cos \theta, \\
V_{++}^1(\mathbf{p}_f) &= \eta_+^\dagger \hat{V}^1(\mathbf{p}_f) \eta_+ = (10) E_v \begin{pmatrix} \frac{-e^{-i\phi} \sin \theta}{-1 + \cos \theta} & 1 - \cos \theta \\ -1 + \cos \theta & -e^{i\phi} \sin \theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -e^{-i\phi} E_v \sin \theta, \\
V_{++}^2(\mathbf{p}_f) &= \eta_+^\dagger \hat{V}^2(\mathbf{p}_f) \eta_+ = (10)i E_v \begin{pmatrix} -e^{-i\phi} \sin \theta & -1 + \cos \theta \\ -1 + \cos \theta & e^{i\phi} \sin \theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -i E_v e^{-i\phi} \sin \theta, \\
V_{++}^3(\mathbf{p}_f) &= \eta_+^\dagger \hat{V}^3(\mathbf{p}_f) \eta_+ = (10) E_v \begin{pmatrix} -(1 + \cos \theta) & e^{-i\phi} \sin \theta \\ -e^{i\phi} \sin \theta & -(1 + \cos \theta) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -(1 + \cos \theta) E_v; \\
A_{++}^0(\mathbf{p}_f) &= \eta_+^\dagger \hat{A}^0(\mathbf{p}_f) \eta_+ = (10) E_v \begin{pmatrix} \frac{-(1 + \cos \theta)}{-e^{i\phi} \sin \theta} & -e^{-i\phi} \sin \theta \\ -e^{i\phi} \sin \theta & (1 + \cos \theta) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - (1 + \cos \theta) E_v, \\
A_{++}^1(\mathbf{p}_f) &= \eta_+^\dagger \hat{A}^1(\mathbf{p}_f) \eta_+ = (10) \begin{pmatrix} (E_p - m)e^{-i\phi} \sin \theta & (E_p + m) - (E_p - m) \cos \theta \\ (E_p + m) - (E_p - m) \cos \theta & -(E_p - m)e^{i\phi} \sin \theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= (E_p - m)e^{-i\phi} \sin \theta, \\
A_{++}^2(\mathbf{p}_f) &= \eta_+^\dagger \hat{A}^2(\mathbf{p}_f) \eta_+ = (10)i \begin{pmatrix} (E_p - m)e^{-i\phi} \sin \theta & -(E_p + m) + (E_p - m) \cos \theta \\ (E_p + m) - (E_p - m) \cos \theta & (E_p - m)e^{i\phi} \sin \theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= i(E_p - m)e^{-i\phi} \sin \theta, \\
A_{++}^3(\mathbf{p}_f) &= \eta_+^\dagger \hat{A}^3(\mathbf{p}_f) \eta_+ = (10) \begin{pmatrix} (E_p + m) + (E_p - m) \cos \theta & (E_p - m)e^{-i\phi} \sin \theta \\ (E_p - m)e^{i\phi} \sin \theta & -(E_p + m) - (E_p - m) \cos \theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= (E_p + m) + (E_p - m) \cos \theta.
\end{aligned}$$

These components of the nucleon current for all possible spin projections of the initial and final nucleons on the direction of motion of the initial lepton (σ_3 basis) can be assembled into compact expressions for the vector current

$$\begin{aligned}
V_{++}^\eta &= \{E_p + m + \cos \theta (E_p - m), -\sin \theta e^{-i\phi} E_v, -i \sin \theta e^{-i\phi} E_v, -(1 + \cos \theta) E_v\}, \\
V_{--}^\eta &= \{E_p + m + \cos \theta (E_p - m), -\sin \theta e^{+i\phi} E_v, +i \sin \theta e^{+i\phi} E_v, -(1 + \cos \theta) E_v\}, \\
V_{+-}^\eta &= \{-\sin \theta e^{-i\phi} (E_p - m), +(1 - \cos \theta) E_v, -i(1 - \cos \theta) E_v, +\sin \theta e^{-i\phi} E_v\}, \\
V_{-+}^\eta &= \{+\sin \theta e^{+i\phi} (E_p - m), -(1 - \cos \theta) E_v, -i(1 - \cos \theta) E_v, -\sin \theta e^{+i\phi} E_v\},
\end{aligned} \tag{291}$$

and similarly, for the axial-vector nucleon current

$$\begin{aligned}
 A_{++}^\eta &= \{-(1 + \cos \theta)E_\nu, +\sin \theta e^{-i\phi}(E_p - m), \\
 &i \sin \theta e^{-i\phi}(E_p - m), +(E_p + m + \cos \theta(E_p - m))\}, \\
 A_{--}^\eta &= \{+(1 + \cos \theta)E_\nu, -\sin \theta e^{+i\phi}(E_p - m), \\
 &i \sin \theta e^{+i\phi}(E_p - m), -(E_p + m + \cos \theta(E_p - m))\}, \\
 A_{+-}^\eta &= \{-\sin \theta e^{-i\phi}E_\nu, E_p + m - \cos \theta(E_p - m), \\
 &i(E_p + m - \cos \theta(E_p - m)), \sin \theta e^{-i\phi}(E_p - m)\}, \\
 A_{-+}^\eta &= \{-\sin \theta e^{+i\phi}E_\nu, E_p + m - \cos \theta(E_p - m), \\
 &+i(E_p + m - \cos \theta(E_p - m)), \sin \theta e^{+i\phi}(E_p - m)\}.
 \end{aligned} \quad (292)$$

Here the first index $r' = \mp 1$ corresponds to the projection of the initial nucleon (doubled) spin on the direction of motion of the initial lepton, and the second index $r = \mp 1$ corresponds to the projection of the initial nucleon (doubled) spin on the same direction.

10.4.4. σ_3 Basis: Properties of nucleon currents.

Note that expressions for nucleon weak currents obtained in the helicity basis in the form (270) appreciably differ from similar expressions in the σ_3 basis given by formulas (291) and (292). This is caused by the difference in the physical and mathematical meaning of the eigenvalues r, r' .

For example, in expression (270) r, r' are (doubled) projections of the nucleon spin on the initial and final directions of the nucleon momentum, while in (291) and (292) r, r' are (doubled) projections of the nucleon spin on a prefixed axis chosen along the direction of motion (three-momentum) of the initial lepton.

Indeed, if we assume that the interaction of the nucleon results in its backscattering ($\cos \theta = -1$), indices $(++)$ in the helicity basis will mean that the nucleon spin direction changes to the strictly opposite one (together with the direction of its momentum). In the basis where the nucleon spin is quantized along the fixed axis (σ_3 basis), as defined in (285), indices $(++)$ correspond to the situation where the nucleon spin does not change at all its orientation along the chosen axis.

Let us discuss in more detail the results presented in (291) and (292). To this end, we will take a look at the most interesting cases: (i) forward scattering of the nucleon ($\cos \theta = 1$), (ii) backward scattering of the nucleon ($\cos \theta = -1$), and (iii) nonrelativistic regime, when $\sqrt{s} \rightarrow m$.

(i) According to (291), at $\cos \theta = 1$ (and $\sin \theta = 0$) the vector current

$$\lim_{\cos \theta \rightarrow 1} V_{++}^\eta = \lim_{\cos \theta \rightarrow 1} V_{--}^\eta = 2(E_p, 0, 0, -|\mathbf{p}_p|)$$

is reduced to the four-momentum of the nucleon moving along the direction of the initial lepton.

Nucleon spin flip due to this vector current is impossible because

$$\lim_{\cos \theta \rightarrow 1} V_{+-}^\eta = \lim_{\cos \theta \rightarrow 1} V_{-+}^\eta = 0.$$

Conversely, nucleon spin flip becomes possible in forward scattering due to the axial current, because nonzero components of the axial current arise in the plane perpendicular to the lepton motion

$$\begin{aligned}
 \lim_{\cos \theta \rightarrow 1} A_{+-}^\eta &= 2m(0, 1, -i, 0), \\
 \lim_{\cos \theta \rightarrow 1} A_{-+}^\eta &= 2m(0, 1, i, 0).
 \end{aligned}$$

There is complete compensation of the axial currents with opposite nucleon spin projections

$$\lim_{\cos \theta \rightarrow 1} (A_{++}^\eta + A_{--}^\eta) = 0. \quad (293)$$

This cancelation can be understood if one takes into account that the axial current of the fermion with the same initial and final momentum p and the same spin projection r is proportional to the vector of the four-spin s^μ , $\bar{u}(p, r)\gamma^\mu\gamma^5 u(p, r) = 2mrs^\mu$. Therefore,

$$\begin{aligned}
 &\bar{u}(p, +1)\gamma^\mu\gamma^5 u(p, +1) + \bar{u}(p, -1)\gamma^\mu\gamma^5 u(p, -1) \\
 &= 2m(+1)s^\mu + 2m(-1)s^\mu = 0,
 \end{aligned}$$

which exactly corresponds to (293) in the case of the forward scattering of the nucleon.

(ii) At $\cos \theta = -1$ (and $\sin \theta = 0$) the vector current preserving the nucleon spin projection is reduced to

$$\lim_{\cos \theta \rightarrow -1} V_{++}^\eta = \lim_{\cos \theta \rightarrow -1} V_{--}^\eta = 2m(1, 0, 0, 0).$$

Nucleon spin flip is also possible

$$\begin{aligned}
 \lim_{\cos \theta \rightarrow -1} V_{+-}^\eta &= 2|\mathbf{p}_p|(0, 1, -i, 0), \\
 \lim_{\cos \theta \rightarrow -1} V_{-+}^\eta &= 2|\mathbf{p}_p|(0, -1, -i, 0).
 \end{aligned}$$

The axial current preserving projection of the spin is reduced to

$$\begin{aligned}
 \lim_{\cos \theta \rightarrow -1} A_{++}^\eta &= 2m(0, 0, 0, 1), \\
 \lim_{\cos \theta \rightarrow -1} A_{--}^\eta &= 2m(0, 0, 0, -1).
 \end{aligned}$$

Nucleon spin flip is also allowed

$$\begin{aligned}
 \lim_{\cos \theta \rightarrow -1} A_{+-}^\eta &= 2E_p(0, 1, -i, 0), \\
 \lim_{\cos \theta \rightarrow -1} A_{-+}^\eta &= 2E_p(0, 1, i, 0).
 \end{aligned}$$

(iii) Finally, in the nonrelativistic case, the vector current preserves the nucleon spin direction because

$$\begin{aligned}
 \lim_{\sqrt{s} \rightarrow m} V_{++}^\eta &= \lim_{\sqrt{s} \rightarrow m} V_{--}^\eta = 2m(1, 0, 0, 0), \\
 \lim_{\sqrt{s} \rightarrow m} V_{+-}^\eta &= \lim_{\sqrt{s} \rightarrow m} V_{-+}^\eta = 0.
 \end{aligned} \quad (294)$$

The axial current can both preserve the nucleon spin projection due to the terms

$$\lim_{\sqrt{s} \rightarrow m} A_{++}^\eta = 2m(0, 0, 0, 1),$$

$$\lim_{\sqrt{s} \rightarrow m} A_{--}^\eta = 2m(0, 0, 0, -1),$$

and change the nucleon spin projection to the opposite one

$$\lim_{\sqrt{s} \rightarrow m} A_{+-}^\eta = 2m(0, 1, -i, 0),$$

$$\lim_{\sqrt{s} \rightarrow m} A_{-+}^\eta = 2m(0, 1, i, 0). \quad (295)$$

Note that similar to (293), in this limit there is reduction

$$\lim_{\sqrt{s} \rightarrow m} (A_{++}^\eta + A_{--}^\eta) = 0, \quad (296)$$

from which it follows that in the coherent term of cross section (119) the contribution from the axial currents is reduced for spin-zero nuclei. A more accurate statement can be obtained from consideration of total expressions (292) for axial currents. Indeed,

$$A_{++}^\eta + A_{--}^\eta = -2i \sin \theta (E_p - m)(0, \sin \phi, \cos \phi, 0)$$

$$\simeq -i \frac{k_0^2}{m} \sin \theta (0, \sin \phi, \cos \phi, 0) \quad (297)$$

$$\simeq -i(1 \text{ MeV}) \frac{k_0^2}{(30 \text{ MeV})^2} \sin \theta (0, \sin \phi, \cos \phi, 0).$$

It is seen that in the general case this four-vector is different from zero if the neutrino energy in the lab system is not zero and the scattering angle is $\theta \neq 0$ or π .

σ_3 Basis: Total V–A nucleon current

In terms of the Dirac spinors of the initial and final nucleons, components of the weak nucleon ($V-A$) current are defined as previously,

$$h_{\pm\pm}^\mu(\mathbf{p}_f) g_V \bar{u}(\mathbf{p}_f, \pm) \gamma^\mu u(\mathbf{p}_i, \pm) - g_A \bar{u}(\mathbf{p}_f, \pm) \gamma^\mu \gamma_5 u(\mathbf{p}_i, \pm)$$

$$= g_V V_{\pm\pm}^{\eta\mu}(\mathbf{p}_f) - g_A A_{\pm\pm}^{\eta\mu}(\mathbf{p}_f).$$

For four possible combinations of spin projections of the final and initial nucleons on the direction of the three-momentum of the initial lepton there are, according to (291) and (292), the following explicit expressions for components of the weak nucleon current

$$h_{++}^\mu = \{g_A E_v (1 + \cos \theta) + g_V (E_p + m + (E_p - m) \cos \theta),$$

$$-e^{-i\phi} (E_p g_A + E_v g_V - g_A m) \sin \theta,$$

$$-ie^{-i\phi} (E_p g_A + E_v g_V - g_A m) \sin \theta,$$

$$-E_v g_V (1 + \cos \theta) - g_A (E_p + m + (E_p - m) \cos \theta)\};$$

$$h_{--}^\mu = \{-g_A E_v (1 + \cos \theta)$$

$$+ g_V (E_p + m + (E_p - m) \cos \theta),$$

$$e^{i\phi} (E_p g_A - E_v g_V - g_A m) \sin \theta,$$

$$ie^{i\phi} (-E_p g_A + E_v g_V + g_A m) \sin \theta,$$

$$-E_v g_V + g_A (E_p + m)$$

$$+ (E_p g_A - E_v g_V - g_A m) \cos \theta\}; \quad (298)$$

$$h_{+-}^\mu = \{e^{-i\phi} (E_v g_A + g_V (-E_p + m)) \sin \theta,$$

$$g_V (E_v - E_v \cos \theta)$$

$$- g_A (E_p + m + (-E_p + m) \cos \theta),$$

$$i(E_v g_V (-1 + \cos \theta)$$

$$+ g_A (E_p + m + (-E_p + m) \cos \theta)),$$

$$e^{-i\phi} (-E_p g_A + E_v g_V + g_A m) \sin \theta\};$$

$$h_{-+}^\mu = \{e^{i\phi} (E_v g_A + g_V (E_p - m)) \sin \theta,$$

$$E_v g_V (-1 + \cos \theta) - g_A (E_p + m + (-E_p + m) \cos \theta),$$

$$i(E_v g_V (-1 + \cos \theta) - g_A (E_p + m + (-E_p + m) \cos \theta)),$$

$$-e^{i\phi} (E_p g_A + E_v g_V - g_A m) \sin \theta\}.$$

These components are used below for calculating scalar products of the leptonic and nucleon currents (l, h).

10.4.5. σ_3 Basis: Scalar products of currents (l, h).

Recall that scalar products of the leptonic and nucleon four-currents are defined by the convolution with the metric $g_{\mu\nu}$ tensor, $(l_{\nu/\bar{\nu}} \cdot h_r^\eta) = l_{(\nu/\bar{\nu})}^\mu g_{\mu\nu} h_r^{\eta\nu}$, where components of the nucleon four-current are given by equations (298), and the neutrino and antineutrino four-currents (as in the helicity basis) have the form (287) and (288)

$$l_\nu = 4E_\nu \left(\cos \frac{\theta}{2}, e^{i\phi} \sin \frac{\theta}{2}, -ie^{i\phi} \sin \frac{\theta}{2}, \cos \frac{\theta}{2} \right),$$

$$l_{\bar{\nu}} = 4E_\nu \left(\cos \frac{\theta}{2}, e^{-i\phi} \sin \frac{\theta}{2}, ie^{-i\phi} \sin \frac{\theta}{2}, \cos \frac{\theta}{2} \right).$$

Considering the definition of the energy of the lepton and the nucleon in their center-of-mass system

$$E_\nu = \frac{s - m^2}{2\sqrt{s}}, \quad E_p = \frac{s + m^2}{2\sqrt{s}},$$

and relations $g_V = g_L + g_R$, $g_A = g_L - g_R$,

after a number of calculations the following expressions for the neutrino–nucleon scalar products in the σ_3 basis are obtained:³⁵

³⁵In [19] only the neutrino case was considered. Unfortunately, in formula (C33), $(l, h_{++}^\eta) = 8(s - m^2) \cos \frac{\theta}{2} \times$
 $\left(g_L - g_R \sin^2 \frac{\theta}{2} \frac{m}{\sqrt{s}} (1 - \frac{m}{\sqrt{s}}) \right)$, an unhappy misprint was made
 (the sign – of g_R , cf. (299)).

$$\begin{aligned}
(l_\nu \cdot h_{-}^\eta) &= 8(s - m^2) \cos \frac{\theta}{2} g_R \left(1 - \left(1 - \frac{m}{\sqrt{s}} \right) \sin^2 \frac{\theta}{2} \right) \\
&= 8(s - m^2) \cos \frac{\theta}{2} g_R \frac{(m + \sqrt{s} + (-m + \sqrt{s}) \cos \theta)}{2\sqrt{s}}, \\
(l_\nu \cdot h_{++}^\eta) &= 8(s - m^2) \\
&\quad \times \cos \frac{\theta}{2} \left(g_L + g_R \frac{m}{\sqrt{s}} \left(1 - \frac{m}{\sqrt{s}} \right) \sin^2 \frac{\theta}{2} \right) \\
&= 8(s - m^2) \cos \frac{\theta}{2} \left(g_L + \frac{g_R m (-m + \sqrt{s}) \sin^2 \frac{\theta}{2}}{s} \right), \quad (299) \\
(l_\nu \cdot h_{-+}^\eta) &= 8e^{i\varphi} (s - m^2) \\
&\quad \times \sin \frac{\theta}{2} \left(g_L - g_R \frac{m}{\sqrt{s}} \left(1 - \left(1 - \frac{m}{\sqrt{s}} \right) \sin^2 \frac{\theta}{2} \right) \right), \\
(l_{\bar{\nu}} \cdot h_{+-}^\eta) &= -8(s - m^2) e^{-i\varphi} g_R \left(1 - \frac{m}{\sqrt{s}} \right) \sin \frac{\theta}{2} \cos^2 \frac{\theta}{2}.
\end{aligned}$$

Likewise, for the antineutrino–nucleon scalar products one gets

$$\begin{aligned}
(l_{\bar{\nu}} \cdot h_{-}^\eta) &= 8(s - m^2) \\
&\quad \times \cos \frac{\theta}{2} \left(g_R + \frac{g_L m (\sqrt{s} - m) \sin^2 \frac{\theta}{2}}{s} \right) \\
&= 8(s - m^2) \cos \frac{\theta}{2} \left(g_R + g_L \frac{m}{\sqrt{s}} \left(1 - \frac{m}{\sqrt{s}} \right) \sin^2 \frac{\theta}{2} \right), \\
(l_{\bar{\nu}} \cdot h_{++}^\eta) &= 8(s - m^2) e^{i\varphi} g_L \left(1 - \frac{m}{\sqrt{s}} \right) \sin \frac{\theta}{2} \cos^2 \frac{\theta}{2}, \quad (300) \\
(l_{\bar{\nu}} \cdot h_{-+}^\eta) &= -8(s - m^2) e^{-i\varphi} \\
&\quad \times \sin \frac{\theta}{2} \left(g_R - g_L \frac{m}{\sqrt{s}} \left(1 - \left(1 - \frac{m}{\sqrt{s}} \right) \sin^2 \frac{\theta}{2} \right) \right), \\
(l_{\bar{\nu}} \cdot h_{+-}^\eta) &= 8(s - m^2) g_L \cos \frac{\theta}{2} \left(1 - \left(1 - \frac{m}{\sqrt{s}} \right) \sin^2 \frac{\theta}{2} \right).
\end{aligned}$$

From the comparison of scalar products (299) and (300) follows a formal rule: in going from the neutrino to the antineutrino, in the formulas for the neutrino it is necessary to replace g_L with g_R and simultaneously g_R with g_L , and also change the signs of the nucleon spin projections from + to – and simultaneously from – to +.

It is worth noting that the scalar products (l, h_{rr}^η) defined, for example, by formulas (299) differ from the scalar products (l, h_{rr}^x) that are given, for example, by relations (277). First, dependence on the azimuth angle ϕ “magically” disappears from the first two expressions (299) that enter into the coherent scattering cross section (the nucleon spin does not change its orientation). In the case of the scalar product (l, h_{++}^η) this takes place because the exponential factor $e^{i\varphi}$ of the neutrino current from (287) is canceled with the corresponding factor $e^{-i\varphi}$ appearing in the nucleon vector V_{++} and axial A_{++} currents from (291) and (292).

In the case of the scalar product (l, h_{-}^η) this takes place because the vector and axial-vector contributions, both now depending on $e^{i2\phi}$, are exactly canceled out since they enter into the final result with the opposite signs.

Second, for the forward scattering, $\theta \rightarrow 0$, the sum of two first expressions from (299) does not depend on the axial coupling constant of the nucleon g_A

$$\lim_{\sqrt{s} \rightarrow m} ((l, h_{++}^\eta) + (l, h_{-}^\eta)) = 8(s - m^2) g_V,$$

and the other two expressions in (299) corresponding to the nucleon spin flip vanish. Further, both scalar products (l, h_{rr}^η) vanish if the neutrino scatters backward ($\theta = \pi$) in the center-of-mass system, since this situation corresponds to the change of the total spin of the neutrino–nucleon system by unity. This observation explains why the coherent cross section vanishes when the recoil nucleus has the maximum kinetic energy.

Finally, in the nonrelativistic limit there is

$$\begin{aligned}
\lim_{\sqrt{s} \rightarrow m} \frac{(l, h_{++}^\eta)}{8(s - m^2)} &= \cos \frac{\theta}{2} g_L, \\
\lim_{\sqrt{s} \rightarrow m} \frac{(l, h_{-}^\eta)}{8(s - m^2)} &= \cos \frac{\theta}{2} g_R, \\
\lim_{\sqrt{s} \rightarrow m} \frac{(l, h_{+-}^\eta)}{8(s - m^2)} &= 0, \\
\lim_{\sqrt{s} \rightarrow m} \frac{(l, h_{-+}^\eta)}{8(s - m^2)} &= e^{i\varphi} \sin \frac{\theta}{2} g_A.
\end{aligned} \quad (301)$$

The factor $8(s - m^2)$ is transferred to the denominator because in this very form it is canceled in the corresponding cross sections. Of interest are two limits for two lower expressions in (301) corresponding to the nucleon spin flip. In the nonrelativistic limit the spin-changing vector currents become zero, as is evident from (294), and only transverse components of axial currents (295) survive. However, the axial current A_{+-}^η in (295) turns out to be orthogonal to the neutrino current when $\sqrt{s} \rightarrow m$. At the same time, its scalar product with A_{-+}^η from (295) is not zero and is proportional to the axial coupling constant g_A .

10.4.6. Relations between the σ_3 basis and the helicity basis. The eigenvectors $\chi_\pm$ and $\eta_\pm$ defined for the initial and final nucleons as (247) in the helicity basis and as (285) in the σ_3 basis are related to each other by the linear transformation

$$\begin{aligned}
\begin{pmatrix} \eta_+(\mathbf{n}_p) \\ \eta_-(\mathbf{n}_p) \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ (\pm)1 & 0 \end{pmatrix} \begin{pmatrix} \chi_+(\mathbf{n}_p) \\ \chi_-(\mathbf{n}_p) \end{pmatrix}; \\
\begin{pmatrix} \eta_+(\mathbf{n}_{p'}) \\ \eta_-(\mathbf{n}_{p'}) \end{pmatrix} &= \begin{pmatrix} \sin \frac{\theta}{2} & \cos \frac{\theta}{2} e^{i\varphi} \\ -\cos \frac{\theta}{2} e^{-i\varphi} & \sin \frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \chi_+(\mathbf{n}_{p'}) \\ \chi_-(\mathbf{n}_{p'}) \end{pmatrix}.
\end{aligned} \quad (302)$$

Correctness of this transformation for the initial particles is easily verified by direct substitution. According to (246) and (286), if the initial states are chosen in the form

$$\chi_+(\mathbf{p}_i) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \chi_-(\mathbf{p}_i) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$\eta_-(\mathbf{p}_i) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \eta_+(\mathbf{p}_i) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

the top line of formula (302) with the plus sign is immediately obtained,

$$\begin{pmatrix} \eta_+(\mathbf{p}_i) \\ \eta_-(\mathbf{p}_i) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ +1 & 0 \end{pmatrix} \begin{pmatrix} \chi_+(\mathbf{p}_i) \\ \chi_-(\mathbf{p}_i) \end{pmatrix}.$$

If the initial eigenstates are chosen with the minus sign,

$$\chi_+(\mathbf{p}_i) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad \chi_-(\mathbf{p}_i) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$\eta_-(\mathbf{p}_i) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \eta_+(\mathbf{p}_i) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

it is clearly seen that the top line of formula (302) with the minus sign is obtained. If the final eigenvectors of the nucleon in these bases are chosen in the form

$$\eta_-(\mathbf{p}_f) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \eta_+(\mathbf{p}_f) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$\chi_-(\mathbf{p}_f) = \begin{pmatrix} e^{-i\phi} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \end{pmatrix}, \quad \chi_+(\mathbf{p}_f) = \begin{pmatrix} \sin \frac{\theta}{2} \\ -e^{i\phi} \cos \frac{\theta}{2} \end{pmatrix},$$

the expression from the bottom line of (302) is valid, since direct substitution yields

$$\begin{aligned} & \sin \frac{\theta}{2} \chi_+(\mathbf{p}_f) + e^{i\phi} \cos \frac{\theta}{2} \chi_-(\mathbf{p}_f) \\ &= \sin \frac{\theta}{2} \begin{pmatrix} \sin \frac{\theta}{2} \\ -e^{i\phi} \cos \frac{\theta}{2} \end{pmatrix} + e^{i\phi} \cos \frac{\theta}{2} \begin{pmatrix} e^{-i\phi} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \end{pmatrix} \\ &= \begin{pmatrix} \sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2} \\ (e^{i\phi} - e^{i\phi}) \sin \frac{\theta}{2} \cos \frac{\theta}{2} \end{pmatrix} \equiv \eta_+(\mathbf{p}_f) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ & \sin \frac{\theta}{2} \chi_-(\mathbf{p}_f) - e^{-i\phi} \cos \frac{\theta}{2} \chi_+(\mathbf{p}_f) \\ &= \sin \frac{\theta}{2} \begin{pmatrix} e^{-i\phi} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \end{pmatrix} - e^{-i\phi} \cos \frac{\theta}{2} \begin{pmatrix} \sin \frac{\theta}{2} \\ -e^{i\phi} \cos \frac{\theta}{2} \end{pmatrix} \\ &= \begin{pmatrix} e^{-i\phi} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \\ \sin^2 \frac{\theta}{2} \end{pmatrix} - \begin{pmatrix} e^{-i\phi} \cos \frac{\theta}{2} \sin \frac{\theta}{2} \\ -e^{i\phi} e^{-i\phi} \cos \frac{\theta}{2} \cos \frac{\theta}{2} \end{pmatrix} \\ &\equiv \eta_-(\mathbf{p}_f) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned}$$

Scalar Products in the σ_3 Basis and the Helicity Basis

The formula of the transition from one basis to the other (302) allows relating the expressions for the nucleon currents and scalar products (l, h) obtained in different bases³⁶

$$(l, h_{++}^\eta) = \sin \frac{\theta}{2} (l, h_{+-}^\chi) + \cos \frac{\theta}{2} e^{-i\phi} (l, h_{--}^\chi),$$

$$(l, h_{+-}^\eta) = -\cos \frac{\theta}{2} e^{i\phi} (l, h_{+-}^\chi) + \sin \frac{\theta}{2} (l, h_{--}^\chi),$$

$$(303)$$

$$(l, h_{++}^\eta) = (\mp 1) \left\{ -\sin \frac{\theta}{2} (l, h_{++}^\chi) - \cos \frac{\theta}{2} e^{-i\phi} (l, h_{+-}^\chi) \right\},$$

$$(l, h_{+-}^\eta) = (\mp 1) \left\{ \cos \frac{\theta}{2} e^{i\phi} (l, h_{++}^\chi) - \sin \frac{\theta}{2} (l, h_{+-}^\chi) \right\}.$$

$$(304)$$

Here the sign (∓ 1) corresponds to the sign (± 1) in the top formula of (302). As a physical interpretation of these formulas, we note that, for example, nucleon currents from the left-hand side of relations (303) correspond to the scattering where the nucleon spin projection on the given quantization axis (direction of initial neutrino) does not change (current h_{++}^η) or changes to the opposite one (current h_{+-}^η). These nucleon currents can be uniquely presented as a linear combination of the current with negative helicity in the initial and final states, h_{--}^χ , and the current that changes the initial negative helicity to the final positive one, h_{+-}^χ .

To prove relations (303) and (304), we calculate their right-hand sides and compare them with the left-hand ones. Thus, the scalar products on the left obtained in the σ_3 basis (299) and divided for convenience by the common factor $8(s - m^2)$ have the form

$$(l, h_{++}^\eta) = \cos \frac{\theta}{2} \left(g_L + g_R \frac{m}{\sqrt{s}} \left(1 - \frac{m}{\sqrt{s}} \right) \sin^2 \frac{\theta}{2} \right),$$

$$(l, h_{+-}^\eta) = \sin \frac{\theta}{2} \left(g_L - g_R \frac{m}{\sqrt{s}} \left(1 - \left(1 - \frac{m}{\sqrt{s}} \right) \sin^2 \frac{\theta}{2} \right) \right),$$

$$(l, h_{+-}^\eta) = -e^{-i\phi} g_R \left(1 - \frac{m}{\sqrt{s}} \right) \sin \frac{\theta}{2} \cos^2 \frac{\theta}{2},$$

$$(l, h_{--}^\eta) = \cos \frac{\theta}{2} g_R \left(1 - \left(1 - \frac{m}{\sqrt{s}} \right) \sin^2 \frac{\theta}{2} \right).$$

$$(305)$$

The scalar products on the right-hand sides of (303) and (304) obtained in the helicity basis (278) and divided by the common factor $8(s - m^2)$ are³⁷

³⁶For example, for any structure similar to the current like $Q_{+-}^\eta = \eta_+^\dagger(\mathbf{p}_f) Q \eta_-(\mathbf{p}_i)$ we have in the helicity basis, according to transformation matrices (302), $\left[\sin \frac{\theta}{2} \chi_+^\dagger(\mathbf{p}_f) + e^{-i\phi} \cos \frac{\theta}{2} \chi_-^\dagger(\mathbf{p}_f) \right] Q [(\pm 1) \chi_+(\mathbf{p}_i)]$, from where follows $Q_{+-}^\eta = (\pm 1) \left\{ \sin \frac{\theta}{2} Q_{++}^\chi + e^{-i\phi} \cos \frac{\theta}{2} Q_{+-}^\chi \right\}$, which is equivalent to the top formula in (304).

³⁷In these formulas, $\chi_+(\mathbf{p}_i) = -\chi_+(\mathbf{p}_i)_{\text{here}}$ for the incoming nucleon, as in [19].

$$\begin{aligned}
(l, h_{++}^\chi) &= e^{-i\phi} (1 - \sin^2 \frac{\theta}{2}) g_R, \\
(l, h_{--}^\chi) &= e^{i\phi} \left(g_L - g_R \sin^2 \frac{\theta}{2} \frac{m^2}{s} \right), \\
(l, h_{+-}^\chi) &= \frac{m}{\sqrt{s}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} g_R, \\
(l, h_{-+}^\chi) &= -\frac{m}{\sqrt{s}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} g_R.
\end{aligned} \tag{306}$$

Using (306), we rearrange the right-hand sides of (303) and (304). For the first line in (303) we obtain

$$\begin{aligned}
&\sin \frac{\theta}{2} (l, h_{++}^\chi) + \cos \frac{\theta}{2} e^{-i\phi} (l, h_{--}^\chi) \\
&= \sin \frac{\theta}{2} \left[\frac{m}{\sqrt{s}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} g_R \right] + \cos \frac{\theta}{2} \\
&\quad \times e^{-i\phi} \left[e^{i\phi} \left(g_L - g_R \sin^2 \frac{\theta}{2} \frac{m^2}{s} \right) \right] \\
&= \cos \frac{\theta}{2} \left(g_L + \frac{m}{\sqrt{s}} \sin^2 \frac{\theta}{2} g_R - g_R \sin^2 \frac{\theta}{2} \frac{m^2}{s} \right) \\
&= \cos \frac{\theta}{2} \left(g_L + g_R \frac{m}{\sqrt{s}} \left(1 - \frac{m}{\sqrt{s}} \right) \sin^2 \frac{\theta}{2} \right) = (l, h_{++}^\eta).
\end{aligned}$$

For the second line in (303) we have

$$\begin{aligned}
&-\cos \frac{\theta}{2} e^{i\phi} (l, h_{++}^\chi) + \sin \frac{\theta}{2} (l, h_{--}^\chi) \\
&= -\cos \frac{\theta}{2} e^{i\phi} \left[\frac{m}{\sqrt{s}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} g_R \right] \\
&\quad + \sin \frac{\theta}{2} \left[e^{i\phi} \left(g_L - g_R \sin^2 \frac{\theta}{2} \frac{m^2}{s} \right) \right] \\
&= -e^{i\phi} \sin \frac{\theta}{2} \frac{m}{\sqrt{s}} \cos^2 \frac{\theta}{2} g_R + \sin \frac{\theta}{2} e^{i\phi} g_L \\
&\quad - \sin \frac{\theta}{2} e^{i\phi} g_R \sin^2 \frac{\theta}{2} \frac{m^2}{s} = e^{i\phi} \sin \frac{\theta}{2} \\
&\quad \times \left(g_L - g_R \frac{m}{\sqrt{s}} \left(1 - \left(1 - \frac{m}{\sqrt{s}} \right) \sin^2 \frac{\theta}{2} \right) \right) = (l, h_{+-}^\eta).
\end{aligned}$$

For the first line in (304) there is

$$\begin{aligned}
&-\sin \frac{\theta}{2} (l, h_{++}^\chi) - \cos \frac{\theta}{2} e^{-i\phi} (l, h_{--}^\chi) \\
&= -\sin \frac{\theta}{2} \left[e^{-i\phi} \left(1 - \sin^2 \frac{\theta}{2} \right) g_R \right] \\
&\quad - \cos \frac{\theta}{2} e^{-i\phi} \left[-\frac{m}{\sqrt{s}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} g_R \right] \\
&= -e^{-i\phi} g_R \sin \frac{\theta}{2} \left(1 - \sin^2 \frac{\theta}{2} \right) + e^{-i\phi} g_R \frac{m}{\sqrt{s}} \\
&\quad \times \sin \frac{\theta}{2} \cos^2 \frac{\theta}{2} = -e^{-i\phi} g_R \left(1 - \frac{m}{\sqrt{s}} \right) \sin \frac{\theta}{2} \cos^2 \frac{\theta}{2} = (l, h_{+-}^\eta).
\end{aligned}$$

For the second line in (304) we have

$$\begin{aligned}
&\cos \frac{\theta}{2} e^{i\phi} (l, h_{++}^\chi) - \sin \frac{\theta}{2} (l, h_{--}^\chi) \\
&= \cos \frac{\theta}{2} e^{i\phi} \left[e^{-i\phi} \left(1 - \sin^2 \frac{\theta}{2} \right) g_R \right] \\
&\quad - \sin \frac{\theta}{2} \left[-\frac{m}{\sqrt{s}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} g_R \right]
\end{aligned}$$

$$\begin{aligned}
&= \cos \frac{\theta}{2} g_R \left(1 - \sin^2 \frac{\theta}{2} \right) \\
&\quad + \sin^2 \frac{\theta}{2} \frac{m}{\sqrt{s}} \cos \frac{\theta}{2} g_R \\
&= \cos \frac{\theta}{2} g_R \left(1 - \left(1 - \frac{m}{\sqrt{s}} \right) \sin^2 \frac{\theta}{2} \right) = (l, h_{+-}^\eta).
\end{aligned}$$

This proves validity of the relations in the form (303) and (304) between the scalar products of currents obtained in different bases. According to the logic in footnote 54, i.e., that dependence on the choice of the basis is only in formulas for rearrangement of nucleon bispinors in the form (302), relations like (303) and (304) are valid for both neutrino–nucleon $(l, h_{r'}^\eta)$, and anti-neutrino–nucleon $(l, h_{r'}^\eta)$ scalar products.

Squares of Scalar Products in the σ_3 Basis and the Helicity Basis

Since final expressions from cross sections of (anti)neutrino–nucleus scattering do not depend on the choice of the basis in which expressions for scalar products are obtained, in practical calculations it is convenient to use the following relations for squares of scalar products:

$$\begin{aligned}
|l, h_{++}^\eta|^2 + |l, h_{--}^\eta|^2 &= |l, h_{++}^\chi|^2 + |l, h_{--}^\chi|^2, \\
|l, h_{+-}^\eta|^2 + |l, h_{-+}^\eta|^2 &= |l, h_{+-}^\chi|^2 + |l, h_{-+}^\chi|^2.
\end{aligned} \tag{307}$$

Their validity can be demonstrated as follows. According to expressions for scalars in the σ_3 basis (305) where the factor $8(s - m^2)$ is everywhere omitted, their squares are

$$\begin{aligned}
|l, h_{++}^\eta|^2 &= \cos^2 \frac{\theta}{2} \left(g_L + g_R \frac{m}{\sqrt{s}} \left(1 - \frac{m}{\sqrt{s}} \right) \sin^2 \frac{\theta}{2} \right)^2, \\
|l, h_{--}^\eta|^2 &= \cos^2 \frac{\theta}{2} g_R^2 \left(1 - \left(1 - \frac{m}{\sqrt{s}} \right) \sin^2 \frac{\theta}{2} \right)^2, \\
|l, h_{+-}^\eta|^2 &= \sin^2 \frac{\theta}{2} \left(g_L - g_R \frac{m}{\sqrt{s}} \left(1 - \left(1 - \frac{m}{\sqrt{s}} \right) \sin^2 \frac{\theta}{2} \right) \right)^2, \\
|l, h_{-+}^\eta|^2 &= g_R^2 \left(1 - \frac{m}{\sqrt{s}} \right)^2 \sin^2 \frac{\theta}{2} \cos^4 \frac{\theta}{2}.
\end{aligned} \tag{308}$$

Likewise, squares of scalars in the helicity basis (306) are³⁸

$$\begin{aligned}
|l, h_{++}^\chi|^2 &= \left(1 - \sin^2 \frac{\theta}{2} \right)^2 g_R^2 = g_R^2 \cos^4 \frac{\theta}{2}, \\
|l, h_{--}^\chi|^2 &= \left(g_L - g_R \sin^2 \frac{\theta}{2} \frac{m^2}{s} \right)^2, \\
|l, h_{+-}^\chi|^2 &= g_R^2 \frac{m^2}{s} \sin^2 \frac{\theta}{2} \cos^2 \frac{\theta}{2}, \\
|l, h_{-+}^\chi|^2 &= g_R^2 \frac{m^2}{s} \sin^2 \frac{\theta}{2} \cos^2 \frac{\theta}{2}.
\end{aligned} \tag{309}$$

³⁸In the squares of the scalars the dependence on $\chi_+(\mathbf{p}_i) = -\chi_+(\mathbf{p}_i)_{\text{mine}}$ disappears.

Considering (308) and (309), sums of squares of scalars from (307) are directly calculated

$$\begin{aligned}
 |(l, h_{-}^{\eta})|^2 + |(l, h_{+}^{\eta})|^2 &= \cos^2 \frac{\theta}{2} g_R^2 \left(1 - \left(1 - \frac{m}{\sqrt{s}}\right) \sin^2 \frac{\theta}{2}\right)^2 \\
 &+ g_R^2 \left(1 - \frac{m}{\sqrt{s}}\right)^2 \sin^2 \frac{\theta}{2} \cos^4 \frac{\theta}{2} = g_R^2 \cos^2 \frac{\theta}{2} \\
 &\times \left[1 - 2 \left(1 - \frac{m}{\sqrt{s}}\right) \sin^2 \frac{\theta}{2} + \left(1 - \frac{m}{\sqrt{s}}\right)^2 \sin^2 \frac{\theta}{2}\right] \\
 &= g_R^2 \cos^2 \frac{\theta}{2} \left[1 - \left(1 - \frac{m^2}{s}\right) \sin^2 \frac{\theta}{2}\right]; \\
 |(l, h_{+}^{\chi})|^2 + |(l, h_{-}^{\chi})|^2 &= g_R^2 \cos^4 \frac{\theta}{2} + g_R^2 \frac{m^2}{s} \sin^2 \frac{\theta}{2} \cos^2 \frac{\theta}{2} \\
 &= g_R^2 \cos^2 \frac{\theta}{2} \left[1 - \sin^2 \frac{\theta}{2} + \frac{m^2}{s} \sin^2 \frac{\theta}{2}\right] \\
 &= |(l, h_{-}^{\eta})|^2 + |(l, h_{+}^{\eta})|^2.
 \end{aligned}$$

The second equality from (307) is proved. A more complicated is the case of the first equation. For simplicity, we introduce provisional notations $a = \sin^2 \frac{\theta}{2}$, $d = (1 - \frac{m}{\sqrt{s}})$, and $g_B = g_R \frac{m}{\sqrt{s}}$, and then the right-hand side of the first equality (307) takes the form

$$\begin{aligned}
 |(l, h_{-}^{\chi})|^2 + |(l, h_{+}^{\chi})|^2 &= \left(g_L - g_R \sin^2 \frac{\theta}{2} \frac{m^2}{s}\right)^2 \\
 &+ g_R^2 \frac{m^2}{s} \sin^2 \frac{\theta}{2} \cos^2 \frac{\theta}{2} = g_L^2 - 2g_L g_R \sin^2 \frac{\theta}{2} \frac{m^2}{s} \\
 &+ g_R^2 \sin^4 \frac{\theta}{2} \frac{m^4}{s^2} + g_R^2 \frac{m^2}{s} \sin^2 \frac{\theta}{2} \left(1 - \sin^2 \frac{\theta}{2}\right) \\
 &= g_L^2 - 2g_L g_R \frac{am^2}{s} + g_R^2 \frac{am^2}{s} \left[\frac{am^2}{s} + (1-a)\right].
 \end{aligned}$$

Let us show that the left-hand side has the same form,

$$\begin{aligned}
 |(l, h_{+}^{\eta})|^2 + |(l, h_{-}^{\eta})|^2 &= + \sin^2 \frac{\theta}{2} \left(g_L - g_R \frac{m}{\sqrt{s}} \left[1 - \left(1 - \frac{m}{\sqrt{s}}\right) \sin^2 \frac{\theta}{2}\right]\right)^2 \\
 &= (1-a)(g_L + g_B da)^2 + a(g_L - g_B(1-da))^2 \\
 &= g_L^2 + 2g_L g_B da + g_B^2 (da)^2 - ag_L^2 - 2g_L g_B da^2 \\
 &\quad - ag_B^2 (da)^2 + ag_L^2 - 2ag_L g_B + 2g_L g_B da^2 \\
 &\quad + ag_B^2 - 2g_B^2 da^2 + ag_B^2 (da)^2 = g_L^2 - 2g_L g_B a[1-d] \\
 &+ g_B^2 [(da)^2 + a - 2da^2] = g_L^2 - 2g_L g_R \frac{m}{\sqrt{s}} a \left[1 - 1 + \frac{m}{\sqrt{s}}\right] \\
 &+ g_R^2 \frac{m^2}{s} \left[a^2 - 2a^2 \frac{m}{\sqrt{s}} + a^2 \frac{m^2}{s} + a - 2a^2 + 2a^2 \frac{m}{\sqrt{s}}\right] \\
 &= g_L^2 - 2g_L g_R \frac{am^2}{s} + g_R^2 \frac{m^2}{s} \left[a^2 \frac{m^2}{s} + a - a^2\right] \\
 &= |(l, h_{-}^{\chi})|^2 + |(l, h_{+}^{\chi})|^2.
 \end{aligned}$$

This proves validity of the first equality from (307).

It follows from (307) that summation over final nucleon spin projections for fixed initial helicities $r = -1$ and $r = +1$ can be performed separately through the corresponding summation of the squares of the scalar products obtained in the σ_3 basis. Indeed, if the helicity of the initial nucleon is fixed, $r = +1(-1)$, the sum over all helicities of the final nucleon ($r' = -1, +1$) is equal to the similar sum over final nucleon spins ($s' = -1, +1$) for the projection of the initial nucleon spin on the fixed axis $s = -1(+1)$,

$$\begin{aligned}
 \sum_{r'=\pm} |(l, h_{r'}^{\chi})|^2 &\equiv |(l, h_{+}^{\chi})|^2 + |(l, h_{-}^{\chi})|^2 \\
 &= |(l, h_{+}^{\eta})|^2 + |(l, h_{-}^{\eta})|^2 \equiv \sum_{s'=\pm} |(l, h_{s'}^{\eta})|^2, \\
 \sum_{r'=\pm} |(l, h_{r'}^{\chi})|^2 &\equiv |(l, h_{+}^{\chi})|^2 + |(l, h_{-}^{\chi})|^2 \\
 &= |(l, h_{+}^{\eta})|^2 + |(l, h_{-}^{\eta})|^2 \equiv \sum_{s'=\pm} |(l, h_{s'}^{\eta})|^2.
 \end{aligned}$$

Generalization to the total sum over all possible nucleon spin projections is

$$\sum_{r', r=\pm} |(l, h_{r'}^{\chi})|^2 = \sum_{s', s=\pm} |(l, h_{s'}^{\eta})|^2. \quad (310)$$

This relation makes it possible to calculate in the σ_3 basis (nucleon spin quantization axis is fixed) both the coherent contribution to the total cross section of the (anti)neutrino–nucleus scattering (for which the σ_3 basis is most adequate) and the incoherent contribution, which is usually considered in the helicity basis.

10.5. Calculation of Cross Sections in Terms of Scalar Products of Currents

Cross sections for scattering of neutrinos and anti-neutrinos by a nucleus (122) are sums of the coherent and incoherent terms

$$\begin{aligned}
 \frac{d\sigma^{v/\bar{v}}}{dT_A} &= \frac{d\sigma_{\text{inc}}^{v/\bar{v}}}{dT_A} + \frac{d\sigma_{\text{coh}}^{v/\bar{v}}}{dT_A} \\
 &\equiv \frac{4G_F^2 m_A}{\pi} \left(\frac{d\tilde{\sigma}_{\text{inc}}^{v/\bar{v}}}{dT_A} + \frac{d\tilde{\sigma}_{\text{coh}}^{v/\bar{v}}}{dT_A} \right), \quad (311)
 \end{aligned}$$

where coherent (124) and incoherent (123) terms are defined in terms of scalar products of currents $(l_{v/\bar{v}}, h_{r', r}^{\eta, f})$ in the σ_3 basis

$$\begin{aligned}
 \frac{d\tilde{\sigma}_{\text{coh}}^{v/\bar{v}}}{dT_A} &= \left| \sum_{f=p, n} \sqrt{g_{\text{coh}}^f} F_f \sum_{k=1}^{A_f} [(l_{v/\bar{v}}, h_{k, +}^{\eta, f}) + (l_{v/\bar{v}}, h_{k, -}^{\eta, f})] \right|^2, \quad (312)
 \end{aligned}$$

$$\frac{d\tilde{\sigma}_{\text{inc}}^{v/\bar{v}}}{dT_A} = \sum_{f=p,n} g_{\text{inc}}^f (1 - |F_f|^2) \times \sum_{k=1}^{A_f} \left[\sum_{r'=\pm} |(l_{v/\bar{v}}, h_{r'+}^{\eta,f})|^2 + \sum_{r'=\pm} |(l_{v/\bar{v}}, h_{r'-}^{\eta,f})|^2 \right]. \quad (313)$$

These expressions are divided into two terms corresponding to the positive and negative (doubled) projection of the initial nucleon spin on the direction of the momentum of the incident neutral massless lepton.

Kinematic definitions and relations

In what follows it will be necessary to go from the variable $\sin^2 \frac{\theta_\star}{2}$ defined in the center-of-mass system of the lepton and the nucleon to the measurable kinematic variables corresponding to the (anti)neutrino–nucleus scattering.

Indeed, since $q^2 \equiv (k - k')^2 = q_{\text{max}}^2 \sin^2 \frac{\theta_\star}{2}$, in the scalar products (l, h) it is possible to replace $\sin^2 \frac{\theta_\star}{2}$ with the ratio $\frac{q^2}{q_{\text{max}}^2}$. As was shown in Section 4, a good approximation can be the relation

$$a \equiv \sin^2 \frac{\theta_\star}{2} \equiv \sin^2 \frac{\theta}{2} = \frac{ys}{s - m^2} = \frac{q^2}{q_{\text{max}}^2} \approx \frac{T_A}{T_A^{\text{max}}}. \quad (314)$$

Here y is the standard Bjorken variable, and s is the square of the total energy in the center-of-mass system of the lepton and the nucleon

$$y = \frac{(p, q)}{(p, k)}, \quad s = (p + k)^2, \quad b^2 = \frac{m_f^2}{s}, \quad (315)$$

where p is the four-momentum of the active nucleon, which is a solution to equation of nuclear energy–momentum conservation (49). In (315), a dimensionless variable b is also introduced, which depends on the nucleon mass $m_f \equiv m$. The auxiliary variables a and b are used in this section to simplify calculations, e.g., in expressions for squares of scalar products from section 6.3 or in formulas (319).

There are the following relations between the variables a, b and y, s, m :

$$\begin{aligned} a(1 - b^2) &= y, \quad ab^2 = \frac{ym^2}{s - m^2}, \quad (1 - a) = 1 - \frac{ys}{s - m^2}, \\ 1 - a + ab^2 &= 1 - a(1 - b^2) = 1 - y, \\ a(1 + b^2) &= y \frac{s + m^2}{s - m^2}, \quad a(1 - b) = \frac{\sqrt{sy}}{\sqrt{s + m}}, \\ a(1 - b)^2 &= y\tau, \quad \tau = \frac{1 - b}{1 + b} = \frac{\sqrt{s - m}}{\sqrt{s + m}}. \end{aligned}$$

Based on the latter relations, useful expressions can be considered for the coherent case

$$\begin{aligned} ab(1 - b) &= \frac{my}{\sqrt{s + m}}, \\ 1 - a(1 - b) &= 1 - \frac{\sqrt{sy}}{\sqrt{s + m}}. \end{aligned} \quad (316)$$

Accordingly, in the incoherent case it is convenient to use relations like

$$\begin{aligned} (1 - a) &= \frac{s(1 - y) - m^2}{s - m^2}, \\ (1 - a)ab^2 &= ym^2 \frac{s(1 - y) - m^2}{(s - m^2)^2}, \\ (1 - a)(1 - a + ab^2) &= \frac{s(1 - y)^2 - m^2(1 - y)}{s - m^2}. \end{aligned} \quad (317)$$

Ultimately, using definitions of $y, s, q^2, E_v = \frac{s - m^2}{2\sqrt{s}}$, and also $\frac{ys}{s - m^2} = \frac{q^2}{q_{\text{max}}^2} \approx \frac{T_A}{T_A^{\text{max}}}$, we can connect y to the nuclear recoil kinetic energy T_A and q^2 by the relation

$$y = \frac{s - m^2}{s} \frac{q^2}{q_{\text{max}}^2} \approx \frac{s - m^2}{s} \frac{T_A}{T_A^{\text{max}}}. \quad (318)$$

10.5.1. Cross section for coherent (anti)neutrino–nucleus scattering in terms of $A_\pm^f$. The summation over the index k (i.e., over the number of f -type nucleons) in (312) allows the coherent cross section to be written in terms of A_+^f and A_-^f , which are equal to the number of the f -type nucleons in the nucleus, which have respective positive and negative spin projections on the quantization axis

$$\frac{d\tilde{\sigma}_{\text{coh}}^{v/\bar{v}}}{dT_A} = \left| \sum_{f=p,n} \sqrt{g_{\text{coh}}^f} F_f [A_+^f(l_{v/\bar{v}}, h_{++}^{\eta,f}) + A_-^f(l_{v/\bar{v}}, h_{--}^{\eta,f})] \right|^2,$$

where $A_\pm^p = Z_\pm$ and $A_\pm^n = N_\pm$ are the number of protons and neutrons in the nucleus before interaction with the spin projection $\pm 1/2$ on the direction of the incident lepton momentum.

Let us use the explicit expressions for the scalar products $(l_{v/\bar{v}}, h_{++}^{\eta,f})$ and $(l_{v/\bar{v}}, h_{--}^{\eta,f})$ from section 6.3 (or appendix 10.4) and the fact that the common factor $8^2(s - m^2)^2$ of all squares of scalar products was already taken into account in the determination of the main dimension cross section factor $\frac{4G_F^2 m_A}{\pi}$ according to approximation (106). Then, for the neutrinos and antineutrinos the respective formulas are obtained

$$\frac{d\tilde{\sigma}_{\text{coh}}^{\nu}}{dT_A} = \cos^2 \frac{\theta}{2} \left| \sum_{f=p,n} \sqrt{g_{\text{coh}}^f} F_f \left\{ A_+^f \left[g_L + g_R \frac{m}{\sqrt{s}} \left(1 - \frac{m}{\sqrt{s}} \right) \sin^2 \frac{\theta}{2} \right] + A_-^f g_R \left[1 - \left(1 - \frac{m}{\sqrt{s}} \right) \sin^2 \frac{\theta}{2} \right] \right\} \right|^2,$$

$$\frac{d\tilde{\sigma}_{\text{coh}}^{\bar{\nu}}}{dT_A} = \cos^2 \frac{\theta}{2} \left| \sum_{f=p,n} \sqrt{g_{\text{coh}}^f} F_f \left\{ A_+^f g_L \left[1 - \left(1 - \frac{m}{\sqrt{s}} \right) \sin^2 \frac{\theta}{2} \right] + A_-^f \left[g_R + g_L \frac{m}{\sqrt{s}} \left(1 - \frac{m}{\sqrt{s}} \right) \sin^2 \frac{\theta}{2} \right] \right\} \right|^2.$$

Or in terms of a and $b = \frac{m_f}{\sqrt{s}}$

$$\frac{d\tilde{\sigma}_{\text{coh}}^{\nu}}{dT_A} = (1-a) \left| \sum_{f=p,n} \sqrt{g_{\text{coh}}^f} F_f \left\{ A_+^f \left[g_L^f + g_R^f b(1-b)a \right] + A_-^f g_R^f \left[1 - (1-b)a \right] \right\} \right|^2,$$

$$\frac{d\tilde{\sigma}_{\text{coh}}^{\bar{\nu}}}{dT_A} = (1-a) \left| \sum_{f=p,n} \sqrt{g_{\text{coh}}^f} F_f \left\{ A_-^f \left[g_R^f + g_L^f b(1-b)a \right] + A_+^f g_L^f \left[1 - (1-b)a \right] \right\} \right|^2.$$

If one opens the square brackets, from the comparison of the expressions

$$\frac{d\tilde{\sigma}_{\text{coh}}^{\nu}}{dT_A} = (1-a) \left| \sum_{f=p,n} \sqrt{g_{\text{coh}}^f} F_f \left\{ A_+^f g_L^f + A_+^f g_R^f b(1-b)a + A_-^f g_R^f \left[1 - (1-b)a \right] \right\} \right|^2,$$

$$\frac{d\tilde{\sigma}_{\text{coh}}^{\bar{\nu}}}{dT_A} = (1-a) \left| \sum_{f=p,n} \sqrt{g_{\text{coh}}^f} F_f \left\{ A_-^f g_R^f + A_-^f g_L^f b(1-b)a + A_+^f g_L^f \left[1 - (1-b)a \right] \right\} \right|^2, \quad (319)$$

it is seen that for obtaining the formula for the anti-neutrino $\frac{d\tilde{\sigma}_{\text{coh}}^{\bar{\nu}}}{dT_A}$ from the formula for neutrino $\frac{d\tilde{\sigma}_{\text{coh}}^{\nu}}{dT_A}$ it is necessary to replace in the latter the factor $A_+^f g_L^f$ with $A_-^f g_R^f$ (and simultaneously $A_-^f g_R^f$ with $A_+^f g_L^f$) and

also replace $A_+^f g_R^f$ with $A_-^f g_L^f$. If $A_+^f = A_-^f$ (i.e., the nucleus has a zero total spin), it suffices to interchange the chiral constants $g_R^f \leftrightarrow g_L^f$.

Expressions (319) can be written in the unified form for the ν -nucleus and $\bar{\nu}$ -nucleus scattering

$$\frac{d\tilde{\sigma}_{\text{coh}}^{\nu/\bar{\nu}}}{dT_A} = (1-a) \left| \sum_{f=p,n} \sqrt{g_{\text{coh}}^f} F_f(q^2) \left\{ A_{\pm}^f \left[g_{L/R}^f + g_{R/L}^f b(1-b)a \right] + A_{\mp}^f g_{R/L}^f \left[1 - (1-b)a \right] \right\} \right|^2$$

$$= \left(1 - \frac{T_A}{T_A^{\text{max}}} \right) \left| \sum_{f=p,n} \sqrt{g_{\text{coh}}^f} F_f \left\{ A_{\pm}^f \left[g_{L/R}^f + g_{R/L}^f \frac{my}{\sqrt{s} + m} \right] + A_{\mp}^f g_{R/L}^f \left[1 - \frac{\sqrt{sy}}{\sqrt{s} + m} \right] \right\} \right|^2, \quad (320)$$

where the left (right) and upper (lower) indices correspond to the neutrino and (anti)neutrino, and the kinematic parameters y and s are already defined in section 10.5.

10.5.2. Cross section of coherent (anti)neutrino-nucleus in terms of A^f and ΔA^f . In the expression for coherent scattering cross section (320) it is possible and, as it turns out, more convenient to go from the pair of nuclear parameters $A_{\pm}$ to another (orthogonal) pair of nuclear parameters, which is a total number of

f -type nucleons A^f and a difference between the number of f -nucleons with the positive and negative spin projections on the preferred axis ось ΔA^f

$$A^f = A_+^f + A_-^f, \quad \Delta A^f = A_+^f - A_-^f;$$

$$\text{and } A_{\pm}^f = \frac{1}{2}(A^f \pm \Delta A^f). \quad (321)$$

This rearrangement is conveniently performed in view of

$$\frac{d\tilde{\sigma}_{\text{coh}}^{\nu}}{dT_A} \propto |A_+^f \alpha_{\nu} + A_-^f \beta_{\nu}|^2 = \left| \frac{A^f + \Delta A^f}{2} \alpha_{\nu} + \frac{A^f - \Delta A^f}{2} \beta_{\nu} \right|^2 = \left| \frac{A^f}{2} (\alpha_{\nu} + \beta_{\nu}) + \frac{\Delta A^f}{2} (\alpha_{\nu} - \beta_{\nu}) \right|^2,$$

$$\frac{d\tilde{\sigma}_{\text{coh}}^{\bar{\nu}}}{dT_A} \propto |A_-^f \alpha_{\bar{\nu}} + A_+^f \beta_{\bar{\nu}}|^2 = \left| \frac{A^f - \Delta A^f}{2} \alpha_{\bar{\nu}} + \frac{A^f + \Delta A^f}{2} \beta_{\bar{\nu}} \right|^2 = \left| \frac{A^f}{2} (\alpha_{\bar{\nu}} + \beta_{\bar{\nu}}) + \frac{(-\Delta A^f)}{2} (\alpha_{\bar{\nu}} - \beta_{\bar{\nu}}) \right|^2, \quad (322)$$

where auxiliary variables are introduced

$$\begin{aligned}\alpha_v &= g_L^f + g_R^f b(1-b)a, \quad \beta_v = g_R^f [1 - (1-b)a]; \\ \alpha_{\bar{v}} &= g_R^f + g_L^f b(1-b)a, \quad \beta_{\bar{v}} = g_L^f [1 - (1-b)a].\end{aligned}\quad (323)$$

From Eqs. (322) it is seen that the problem of the parametrization of cross sections in terms of A^f and ΔA^f is reduced to calculation of sums and differences of the auxiliary variables from (323). Considering kinematic definitions and relations from section 10.5, one obtains

$$\begin{aligned}\alpha_v + \beta_v &= g_L^f + g_R^f [ab(1-b) + 1 - a(1-b)] \\ &= g_L^f + g_R^f + g_R^f [ab(1-b) - a(1-b)] \\ &= g_L^f + g_R^f + g_R^f y \left[\frac{m}{\sqrt{s} + m} - \frac{\sqrt{s}}{\sqrt{s} + m} \right] \\ &= g_L^f + g_R^f - g_R^f y \tau, \\ \alpha_v - \beta_v &= g_L^f + g_R^f [ab(1-b) - 1 + a(1-b)] \\ &= g_L^f - g_R^f + g_R^f [ab(1-b) + a(1-b)] \\ &= g_L^f - g_R^f + g_R^f y \left[\frac{m}{\sqrt{s} + m} + \frac{\sqrt{s}}{\sqrt{s} + m} \right] \\ &= g_L^f - g_R^f + g_R^f y.\end{aligned}$$

Recall that $\tau = \frac{\sqrt{s} - m}{\sqrt{s} + m}$. Similarly (or using replacement $g_L^f \leftrightarrow g_R^f$), we have

$$\begin{aligned}\alpha_{\bar{v}} + \beta_{\bar{v}} &= g_R^f + g_L^f [ab(1-b) + 1 - a(1-b)] \\ &= g_R^f + g_L^f - g_L^f y \tau, \\ \alpha_{\bar{v}} - \beta_{\bar{v}} &= g_R^f + g_L^f [ab(1-b) - 1 + a(1-b)] \\ &= g_R^f - g_L^f + g_L^f y.\end{aligned}$$

Recalling the transition from $g_L^f = \frac{1}{2}(g_V^f + g_A^f)$ and $g_R^f = \frac{1}{2}(g_V^f - g_A^f)$ to $g_V^f = g_L^f + g_R^f$ and $g_A^f = g_L^f - g_R^f$, we can also write the above-obtained relations as

$$\begin{aligned}\alpha_v + \beta_v &= g_V^f - g_R^f y \tau \\ &= g_V^f - \frac{y \tau}{2} (g_V^f - g_A^f) g_V^f \left(1 - \frac{y \tau}{2} \right) + g_A^f \frac{y \tau}{2}, \\ \alpha_v - \beta_v &= g_A^f + g_R^f y \\ &= g_A^f + \frac{y}{2} (g_V^f - g_A^f) g_V^f \frac{y}{2} + g_A^f \left(1 - \frac{y}{2} \right); \\ \alpha_{\bar{v}} + \beta_{\bar{v}} &= g_V^f - g_L^f y \tau \\ &= g_V^f - \frac{y \tau}{2} (g_V^f + g_A^f) g_V^f \left(1 - \frac{y \tau}{2} \right) - g_A^f \frac{y \tau}{2}, \\ \alpha_{\bar{v}} - \beta_{\bar{v}} &= -g_A^f + g_L^f y \\ &= -g_A^f + \frac{y}{2} (g_V^f + g_A^f) g_V^f \frac{y}{2} - g_A^f \left(1 - \frac{y}{2} \right).\end{aligned}$$

Cross sections (320) for coherent neutrino and antineutrino scattering off the nucleus can be expressed in terms of $\alpha_v, \beta_v, \alpha_{\bar{v}}, \beta_{\bar{v}}$, and A^f and ΔA^f using (322) as

$$\begin{aligned}\frac{d\tilde{\sigma}_{\text{coh}}^v}{dT_A} &= (1-a) \left| \sum_{f=p,n} \sqrt{g_{\text{coh}}^f} F_f \left[\frac{A^f}{2} (\alpha_v + \beta_v) + \frac{\Delta A^f}{2} (\alpha_v - \beta_v) \right] \right|^2, \\ \frac{d\tilde{\sigma}_{\text{coh}}^{\bar{v}}}{dT_A} &= (1-a) \left| \sum_{f=p,n} \sqrt{g_{\text{coh}}^f} F_f \left[\frac{A^f}{2} (\alpha_{\bar{v}} + \beta_{\bar{v}}) + \frac{(-\Delta A^f)}{2} (\alpha_{\bar{v}} - \beta_{\bar{v}}) \right] \right|^2.\end{aligned}$$

Hence, considering the expressions obtained for $(\alpha_v \pm \beta_v)$ and $(\alpha_{\bar{v}} \pm \beta_{\bar{v}})$, cross sections for coherent scattering off the nucleus in terms of weak chiral constants have the form

$$\begin{aligned}\frac{d\tilde{\sigma}_{\text{coh}}^v}{dT_A} \Big|_{L,R} &= \frac{(1-a)}{4} \left| \sum_{f=p,n} \sqrt{g_{\text{coh}}^f} F_f \left[A^f (g_L^f + g_R^f (1 - y \tau)) + \Delta A^f (g_L^f - g_R^f (1 - y)) \right] \right|^2, \\ \frac{d\tilde{\sigma}_{\text{coh}}^{\bar{v}}}{dT_A} \Big|_{L,R} &= \frac{(1-a)}{4} \left| \sum_{f=p,n} \sqrt{g_{\text{coh}}^f} F_f \left[A^f (g_R^f + g_L^f (1 - y \tau)) + (-\Delta A^f) (g_R^f - g_L^f (1 - y)) \right] \right|^2.\end{aligned}\quad (324)$$

Cross sections (324) expressed in terms of vector and axial weak constants look like

$$\begin{aligned}\frac{d\tilde{\sigma}_{\text{coh}}^v}{dT_A} \Big|_{V,A} &= \frac{(1-a)}{4} \left| \sum_{f=p,n} \sqrt{g_{\text{coh}}^f} F_f \left[A^f (g_V^f \left(1 - \frac{y \tau}{2} \right) + g_A^f \frac{y \tau}{2} \right) + \Delta A^f \left(g_V^f \frac{y}{2} + g_A^f \left(1 - \frac{y}{2} \right) \right) \right] \right|^2, \\ \frac{d\tilde{\sigma}_{\text{coh}}^{\bar{v}}}{dT_A} \Big|_{V,A} &= \frac{(1-a)}{4} \left| \sum_{f=p,n} \sqrt{g_{\text{coh}}^f} F_f \left[A^f \left(g_V^f \left(1 - \frac{y \tau}{2} \right) - g_A^f \frac{y \tau}{2} \right) + (-\Delta A^f) \left(g_V^f \frac{y}{2} - g_A^f \left(1 - \frac{y}{2} \right) \right) \right] \right|^2.\end{aligned}\quad (325)$$

Further, collecting coefficients of g_V^f and g_A^f , i.e., going from the expression of the type

$$[A^f () + (-\Delta A^f)()] = g_V^f A^f \left(1 - \frac{y \tau}{2} \right) - g_A^f A^f \frac{y \tau}{2} + g_V^f \frac{y}{2} (-\Delta A^f) - g_A^f \left(1 - \frac{y}{2} \right) (-\Delta A^f)$$

to the expression of the type

$$g_V^f \left[A^f \left(1 - \frac{y\tau}{2} \right) + \frac{y}{2} (-\Delta A^f) \right] + (-g_A^f) \left[A^f \frac{y\tau}{2} + \left(1 - \frac{y}{2} \right) (-\Delta A^f) \right],$$

one obtains in accordance with (311) the ultimate formulas for the cross section of the coherent neutrino/antineutrino–nucleus scattering

$$\begin{aligned} \frac{d\sigma_{coh}^{v/\bar{v}}}{dT_A} &= \frac{4G_F^2 m_A}{\pi} \frac{d\sigma_{coh}^{v/\bar{v}}}{dT_A} \\ &= \frac{G_F^2 m_A}{\pi} \left(1 - \frac{T_A}{T_A^{\max}} \right) |G_V^{v/\bar{v}} + G_A^{v/\bar{v}}|^2, \end{aligned} \quad (326)$$

where in the neutrino case

$$\begin{aligned} G_V^v &= \sum_{f=p,n} \sqrt{g_{coh}^f} F_f g_V^f \left(A^f \left[1 - \frac{y\tau}{2} \right] + \Delta A^f \frac{y}{2} \right), \\ G_A^v &= \sum_{f=p,n} \sqrt{g_{coh}^f} F_f g_A^f \left(A^f \frac{y\tau}{2} + \Delta A^f \left[1 - \frac{y}{2} \right] \right), \end{aligned} \quad (327)$$

and in the antineutrino case

$$\begin{aligned} G_V^{\bar{v}} &= \sum_{f=p,n} \sqrt{g_{coh}^f} F_f g_V^f \left[A^f \left(1 - \frac{y\tau}{2} \right) + \frac{y}{2} (-\Delta A^f) \right], \\ G_A^{\bar{v}} &= \sum_{f=p,n} \sqrt{g_{coh}^f} F_f (-g_A^f) \left[A^f \frac{y\tau}{2} + \left(1 - \frac{y}{2} \right) (-\Delta A^f) \right]. \end{aligned} \quad (328)$$

Expression (326) with the coefficients from (327) corresponds up to redefinition of the correction factor g_{coh}^f to the formula for coherent νA scattering (48) from [19].

Other expressions for cross sections of coherent (anti)neutrino–nucleus scattering

It is useful to expand the square of the matrix element in formulas (320) for the coherent (anti)neutrino–nucleus scattering. Introducing temporary designations

$$\begin{aligned} P_f &= \sqrt{g_{coh}^f} F_f A_+^f (g_L^f + g_R^f \alpha), \\ M_f &= \sqrt{g_{coh}^f} F_f A_-^f g_R^f \beta, \end{aligned}$$

where $\alpha = ab(1-b) = \frac{my}{\sqrt{s}+m}$ and

$\beta = (1 - \alpha(1-b)) = -\frac{\sqrt{sy}}{\sqrt{s}+m}$,³⁹ for the “neutrino” formula from (320) one can write

³⁹ A good approximation, when $m \equiv m_p \equiv m_n$.

$$\begin{aligned} \frac{d\sigma_{coh}^v}{dT_A(1-a)} &= \left| \sum_{f=p,n} [P_f + M_f] \right|^2 \\ &= [(P_n + M_n)^* + (P_p + M_p)^*] \\ &\times [(P_n + M_n) + (P_p + M_p)] = |P_n + M_n|^2 + |P_p + M_p|^2 \\ &+ (P_n + M_n)^*(P_p + M_p) + (P_p + M_p)^*(P_n + M_n) \\ &= (P_n + M_n)^2 + (P_p + M_p)^2 \\ &+ 2P_n P_p + 2P_n M_p + 2M_n P_n + 2M_p M_n. \end{aligned}$$

In the last line, a simplifying assumption that P_f and M_f are real values is made without any significant loss of generality. Then, considering that

$$(P_f + M_g)^2 = g_{coh}^f |F_f|^2 [A_+^f g_L^f + g_R^f \alpha + A_-^f g_R^f \beta]^2,$$

we obtain

$$\begin{aligned} \frac{d\sigma_{coh}^v}{dT_A(1-\alpha)} &= \sum_{f=p,n} g_{coh}^f |F_f|^2 \\ &\times [A_+^f (g_L^f + g_R^f \alpha) + A_-^f g_R^f \beta]^2 \\ &+ 2\sqrt{g_{coh}^p g_{coh}^n} F_p F_n [A_+^p A_+^n (g_L^p + g_R^p \alpha)(g_L^n + g_R^n \alpha) \\ &+ A_+^p (g_L^p + g_R^p \alpha) A_-^n g_R^n \beta + A_+^n (g_L^n + g_R^n \alpha) A_-^p g_R^p \beta \\ &+ A_-^p g_R^p \beta A_-^n g_R^n \beta]. \end{aligned}$$

Therefore, the coherent (anti)neutrino–nucleus scattering cross section in terms of g_L^f , g_R^f , and $A_\pm^f$ takes the (cumbersome) form

$$\begin{aligned} \frac{d\sigma_{coh}^{v/\bar{v}}}{dT_A} &= \frac{4G_F^2 m_A}{\pi} \frac{d\sigma_{coh}^{v/\bar{v}}}{dT_A} = \frac{4G_F^2 m_A}{\pi} \left(1 - \frac{T_A}{T_A^{\max}} \right) \\ &\times \sum_{f=p,n} g_{coh}^f |F_f|^2 \left[A_\pm^f \left(g_{L/R}^f + g_{R/L}^f \frac{my}{\sqrt{s}+m} \right) \right. \\ &+ A_\pm^f g_{R/L}^f \left(1 - \frac{\sqrt{sy}}{\sqrt{s}+m} \right) \left. \right]^2 + 2\sqrt{g_{coh}^p g_{coh}^n} F_p F_n \\ &\times \left[A_\pm^p A_\pm^n \left(g_{L/R}^p + g_{R/L}^p \frac{my}{\sqrt{s}+m} \right) \right. \\ &\times \left(g_{L/R}^n + g_{R/L}^n \frac{my}{\sqrt{s}+m} \right) \\ &+ A_\mp^p A_\mp^n g_{L/R}^p g_{R/L}^n \left(1 - \frac{\sqrt{sy}}{\sqrt{s}+m} \right)^2 \\ &+ A_\mp^p A_\mp^n \left(g_{L/R}^p + g_{R/L}^p \frac{my}{\sqrt{s}+m} \right) \\ &\times g_{R/L}^n \left(1 - \frac{\sqrt{sy}}{\sqrt{s}+m} \right) + A_\mp^n A_\mp^p \\ &\times \left(g_{L/R}^p + g_{R/L}^p \frac{my}{\sqrt{s}+m} \right) g_{R/L}^p \left(1 - \frac{\sqrt{sy}}{\sqrt{s}+m} \right) \left. \right]. \end{aligned} \quad (329)$$

A similar expansion of the formula for the coherent (neutrino) scattering in terms of A_f , ΔA_f , g_L , and g_R looks like

$$\begin{aligned} \frac{d\sigma_{\text{coh}}^v}{dT_A} &= \frac{G_F^2 m_A}{\pi} \left(1 - \frac{T_A}{T_A^{\text{max}}}\right) \\ &\times \left\{ \sum_{f=p,n} g_{\text{coh}}^f |F_f|^2 \left[A_f^2 [g_L^f + g_R^f(1-y\tau)]^2 \right. \right. \\ &+ 2A_f \Delta A_f [g_L^f + g_R^f(1-y\tau)] [g_L^f - g_R^f(1-y)] \\ &+ \Delta A_f^2 [g_L^f - g_R^f(1-y)]^2 \left. \right] + 2\sqrt{g_{\text{coh}}^p g_{\text{coh}}^n} F_p F_n \\ &\times A_p A_n [g_L^p + g_R^p(1-y\tau)] [g_L^n + g_R^n(1-y\tau)] \\ &+ \Delta A_p \Delta A_n [g_L^p - g_R^p(1-y)] [g_L^n - g_R^n(1-y)] \\ &+ A_n \Delta A_p [g_L^n + g_R^n(1-y\tau)] [g_L^p - g_R^p(1-y)] \\ &+ A_p \Delta A_n [g_L^p + g_R^p(1-y\tau)] [g_L^n - g_R^n(1-y)] \left. \right\}. \end{aligned}$$

If spins of protons and neutrons are “balanced” (i.e., $\Delta A_f = 0$), from the previous expression one obtains a rather simple formula for the coherent νA scattering

$$\begin{aligned} \frac{d\sigma_{\text{coh}}^v}{dT_A} &= \frac{C_F^2 m_A}{\pi} \left(1 - \frac{T_A}{T_A^{\text{max}}}\right) \\ &\times \left\{ \sum_{f=p,n} \sqrt{g_{\text{coh}}^f} F_f [g_L^f + g_R^f(1-y\tau)]^2 \right\}. \end{aligned}$$

A formula for the coherent $\bar{\nu} A$ scattering cross section for $\Delta A_f = 0$ is obtained from this expression by a simple replacement $g_L^f \leftrightarrow g_R^f$.

It is worth to make a similar expansion for the square of the matrix element for cross sections (325) expressed in terms of g_V^f , g_A^f , A_f , ΔA_f , and $\tau = \frac{\sqrt{s-m}}{\sqrt{s+m}}$. Introduction of the variables

$$\begin{aligned} P_f &= \sqrt{g_{\text{coh}}^f} F_f A_f (g_V^f - g_R^f y \tau), \\ M_f &= \sqrt{g_{\text{coh}}^f} F_f \Delta A_f (g_A^f + g_R^f y \tau), \end{aligned}$$

allows the coherent νA scattering cross section from (325) to be conveniently written, with g_R temporarily kept, as

$$\begin{aligned} \frac{4d\tilde{\sigma}_{\text{coh}}^v}{dT_A} &= \left| \sum_{f=p,n} \sqrt{g_{\text{coh}}^f} F_f [A_f (g_V^f - g_R^f y \tau) \right. \\ &+ \Delta A_f (g_A^f + g_R^f y \tau)] \left. \right|^2 = \left| \sum_{f=p,n} [P_f + M_f] \right|^2 \\ &= (P_n + M_n)^2 + (P_p + M_p)^2 + 2P_n P_p \\ &+ 2P_n M_p + 2M_n P_p + 2M_p M_n, \end{aligned}$$

where $(P_f + M_f)^2 = g_{\text{coh}}^f |F_f|^2 [A_f^2 (g_V^f - g_R^f y \tau)^2 + 2A_f (g_V^f - g_R^f y \tau) \Delta A_f (g_A^f + g_R^f y \tau) + \Delta A_f^2 (g_A^f + g_R^f y \tau)^2]$ and

$$\begin{aligned} &P_n P_p + P_n M_p + M_n P_p + M_p M_n \\ &= \sqrt{g_{\text{coh}}^p g_{\text{coh}}^n} F_p F_n A_p A_n (g_V^p - g_R^p y \tau) (g_V^n - g_R^n y \tau) \\ &+ \sqrt{g_{\text{coh}}^p g_{\text{coh}}^n} F_p F_n A_n \Delta A_p (g_V^p - g_R^p y \tau) (g_A^n + g_R^n y \tau) \\ &+ \sqrt{g_{\text{coh}}^p g_{\text{coh}}^n} F_p F_n A_p \Delta A_n (g_V^p - g_R^p y \tau) (g_A^n + g_R^n y \tau) \\ &+ \sqrt{g_{\text{coh}}^p g_{\text{coh}}^n} F_p F_n \Delta A_p \Delta A_n (g_A^p + g_R^p y \tau) (g_A^n + g_R^n y \tau) \\ &= \sqrt{g_{\text{coh}}^p g_{\text{coh}}^n} F_p F_n \{ A_p A_n (g_V^p - g_R^p y \tau) (g_V^n - g_R^n y \tau) \\ &+ A_n \Delta A_p (g_V^n - g_R^n y \tau) (g_A^p + g_R^p y \tau) + A_n \Delta A_p (g_V^p - g_R^p y \tau) (g_A^n + g_R^n y \tau) \\ &+ \Delta A_p \Delta A_n (g_A^p + g_R^p y \tau) (g_A^n + g_R^n y \tau) \}. \end{aligned}$$

Then this expression for the cross sections is slightly simplified

$$\begin{aligned} \frac{4d\tilde{\sigma}_{\text{coh}}^v}{dT_A(1-\alpha)} &= \sum_{f=p,n} g_{\text{coh}}^f |F_f|^2 [A_f^2 (g_V^f - g_R^f y \tau)^2 \\ &+ 2A_f (g_V^f - g_R^f y \tau) \Delta A_f (g_A^f + g_R^f y \tau) \\ &+ \Delta A_f^2 (g_A^f + g_R^f y \tau)^2] + 2\sqrt{g_{\text{coh}}^p g_{\text{coh}}^n} F_p F_n \\ &\times [A_p A_n (g_V^p - g_R^p y \tau) (g_V^n - g_R^n y \tau) \\ &+ A_n \Delta A_p (g_V^n - g_R^n y \tau) (g_A^p + g_R^p y \tau) \\ &+ A_p \Delta A_n (g_V^p - g_R^p y \tau) (g_A^n + g_R^n y \tau) \\ &+ \Delta A_p \Delta A_n (g_A^p + g_R^p y \tau) (g_A^n + g_R^n y \tau)]. \end{aligned}$$

Ultimately, with g_R excluded by the replacement

$$\begin{aligned} (g_V^f - g_R^f y \tau) &= g_V^f \left(1 - \frac{y\tau}{2}\right) + g_A^f \frac{y\tau}{2}, \\ (g_A^f + g_R^f y \tau) &= g_V^f \frac{y}{2} + g_A^f \left(1 - \frac{y\tau}{2}\right), \end{aligned}$$

the coherent neutrino–nucleus scattering cross section (325) in terms of A_f , ΔA_f , and $g_{V,A}$ takes the final (cumbersome again) form

$$\begin{aligned} \frac{d\sigma_{\text{coh}}^v}{dT_A} &= \frac{G_F^2 m_A}{\pi} \left(1 - \frac{T_A}{T_A^{\text{max}}}\right) \left\{ \sum_{f=p,n} g_{\text{coh}}^f |F_f|^2 \right. \\ &\times \left[A_f^2 \left(g_V^f \left(1 - \frac{y\tau}{2}\right) + g_A^f \frac{y\tau}{2} \right)^2 + 2A_f \Delta A_f \right. \\ &\times \left(g_V^f \left(1 - \frac{y\tau}{2}\right) + g_A^f \frac{y\tau}{2} \right) \left(g_V^f \frac{y}{2} + g_A^f \left(1 - \frac{y\tau}{2}\right) \right) \\ &\left. \left. + \Delta A_f^2 \left(g_V^f \frac{y}{2} + g_A^f \left(1 - \frac{y\tau}{2}\right) \right)^2 \right] \right. \\ &+ 2\sqrt{g_{\text{coh}}^p g_{\text{coh}}^n} F_p F_n \left[A_p A_n \left(g_V^p \left(1 - \frac{y\tau}{2}\right) + g_A^p \frac{y\tau}{2} \right) \right. \\ &\times \left(g_V^n \left(1 - \frac{y\tau}{2}\right) + g_A^n \frac{y\tau}{2} \right) \\ &+ A_n \Delta A_p \left(g_V^n \left(1 - \frac{y\tau}{2}\right) + g_A^n \frac{y\tau}{2} \right) \left(g_V^p \frac{y}{2} + g_A^p \left(1 - \frac{y\tau}{2}\right) \right) \\ &+ A_p \Delta A_n \left(g_V^p \left(1 - \frac{y\tau}{2}\right) + g_A^p \frac{y\tau}{2} \right) \left(g_V^n \frac{y}{2} + g_A^n \left(1 - \frac{y\tau}{2}\right) \right) \\ &\left. \left. + \Delta A_p \Delta A_n \left(g_V^p \frac{y}{2} + g_A^p \left(1 - \frac{y\tau}{2}\right) \right) \left(g_V^n \frac{y}{2} + g_A^n \left(1 - \frac{y\tau}{2}\right) \right) \right] \right\}. \end{aligned}$$

However, if $\Delta A_f = 0$, a simpler formula is obtained for the coherent neutrino–nucleus scattering cross section

$$\begin{aligned} \frac{d\sigma_{\text{coh}}^{\nu}}{dT_A} &= \frac{G_F^2 m_A}{\pi} \left(1 - \frac{T_A}{T_A^{\text{max}}}\right) \left\{ \sum_{f=p,n} g_{\text{coh}}^f |F_f|^2 A_f^2 \right. \\ &\times \left[g_V^f \left(1 - \frac{y\tau}{2}\right) + g_A^f \frac{y\tau}{2} \right]^2 + 2\sqrt{g_{\text{coh}}^p g_{\text{coh}}^n} F_p F_n A_p A_n \\ &\times \left[g_V^p \left(1 - \frac{y\tau}{2}\right) + g_A^p \frac{y\tau}{2} \right] \left[g_V^n \left(1 - \frac{y\tau}{2}\right) + g_A^n \frac{y\tau}{2} \right] \Big\}, \end{aligned}$$

which is further simplified and generalized to the case of coherent $\bar{\nu}A$ scattering (through replacement $g_A \rightarrow -g_A$) in the form

$$\begin{aligned} \frac{d\sigma_{\text{coh}}^{\nu/\bar{\nu}}}{dT_A} &= \frac{G_F^2 m_A}{\pi} \left(1 - \frac{T_A}{T_A^{\text{max}}}\right) \left(\sum_{f=p,n} \sqrt{g_{\text{coh}}^f} F_f A_f \right. \\ &\times \left[g_V^f \left(1 - \frac{y\tau}{2}\right) \pm g_A^f \frac{y\tau}{2} \right]^2 \Big). \end{aligned} \quad (330)$$

10.5.3. Cross section for incoherent (anti)neutrino–nucleus scattering in terms of $A_{\pm}^f$. Summation over the

index k in (313) allows incoherent cross section (313) to be written in terms of $A_{\pm}^f$ and $A_{\mp}^f$

$$\begin{aligned} \frac{d\tilde{\sigma}_{\text{inc}}^{\nu/\bar{\nu}}}{dT_A} &= \sum_{f=p,n} \hat{F}_f \\ &\times \left[A_{\mp}^f \sum_{r'=\pm} |(l_{\nu/\bar{\nu}}, h_{r'-}^{\eta,f})|^2 + A_{\pm}^f \sum_{r'=\pm} |(l_{\nu/\bar{\nu}}, h_{r'+}^{\eta,f})|^2 \right], \end{aligned}$$

where, as above, $A_{\pm}^p = Z_{\pm}$ and $A_{\pm}^n = N_{\pm}$ are the numbers of initial protons and neutrons with the spin projection $\pm 1/2$ on the direction of the incident lepton momentum. For convenient notation, an auxiliary quantity

$$\hat{F}_f(q^2) = g_{\text{inc}}^f (1 - |F_f(q^2)|^2), \quad (331)$$

is temporarily introduced. Further, one should use the expressions for the squares of the scalar products $|(l_{\nu/\bar{\nu}}, h_{r',\pm}^{\eta,f})|^2$ from appendix 10.4⁴⁰. Since there are expressions for their sums (over spins of final nucleons $r' = -1, +1$)

$$\begin{aligned} |(l_{\nu}, h_{-}^{\eta,f})|^2 + |(l_{\nu}, h_{+}^{\eta,f})|^2 &= (1-a)g_R^2 [1 - (1-b)a]^2 + a(1-a)^2 g_R^2 (1-b)^2, \\ |(l_{\nu}, h_{-+}^{\eta,f})|^2 + |(l_{\nu}, h_{+-}^{\eta,f})|^2 &= a(g_L - g_R b [1 - (1-b)a])^2 + (1-a)[g_L + g_R b(1-b)a]^2, \\ |(l_{\bar{\nu}}, h_{-}^{\eta,f})|^2 + |(l_{\bar{\nu}}, h_{+}^{\eta,f})|^2 &= (1-a)[g_R + g_L b(1-b)a]^2 + a(g_R - g_L b [1 - (1-b)a])^2, \\ |(l_{\bar{\nu}}, h_{-+}^{\eta,f})|^2 + |(l_{\bar{\nu}}, h_{+-}^{\eta,f})|^2 &= a(1-a)^2 g_L^2 (1-b)^2 + (1-a)g_L^2 [1 - (1-b)a]^2, \end{aligned}$$

for the incoherent νA and $\bar{\nu} A$ scattering cross sections one consequently obtains

$$\begin{aligned} \frac{d\tilde{\sigma}_{\text{inc}}^{\nu}}{dT_A} &= \sum_{f=p,n} F_f \left[A_{\mp}^f g_R^2 (1-a) \{ [1 - (1-b)a]^2 + a(1-a)(1-b)^2 \} \right. \\ &+ A_{\pm}^f \{ a(g_L - g_R b [1 - (1-b)a])^2 + (1-a)[g_L + g_R b(1-b)a]^2 \} \Big], \\ \frac{d\tilde{\sigma}_{\text{inc}}^{\bar{\nu}}}{dT_A} &= \sum_{f=p,n} F_f \left[A_{\mp}^f g_L^2 (1-a) \{ [1 - (1-b)a]^2 + a(1-a)(1-b)^2 \} \right. \\ &+ A_{\pm}^f \{ a(g_R - g_L b [1 - (1-b)a])^2 + (1-a)[g_R + g_L b(1-b)a]^2 \} \Big]. \end{aligned}$$

Introducing, for convenient intermediate calculations, the auxiliary variables

$$\begin{aligned} X &\equiv \{ [1 - (1-b)a]^2 + a(1-a)(1-b)^2 \} \\ \text{and } Z(P, Q) &\equiv \{ a(P - Qb [1 - (1-b)a])^2 + (1-a)[P + Qb(1-b)a]^2 \}, \end{aligned} \quad (332)$$

⁴⁰For convenience, they are given below in terms of a and b , divided by the factor $64(s - m^2)^2$,

$$\begin{aligned} |(l_{\nu} \cdot h_{-}^{\eta})|^2 &= (1-a)g_R^2 [1 - (1-b)a]^2, \quad |(l_{\bar{\nu}} \cdot h_{-}^{\eta})|^2 = (1-a)[g_R + g_L b(1-b)a]^2, \\ |(l_{\nu} \cdot h_{+}^{\eta})|^2 &= (1-a)g_L^2 [1 - (1-b)a]^2, \quad |(l_{\bar{\nu}} \cdot h_{+}^{\eta})|^2 = (1-a)[g_L + g_L b(1-b)a]^2, \\ |(l_{\nu} \cdot h_{-+}^{\eta})|^2 &= a(g_L - g_R b [1 - (1-b)a])^2, \quad |(l_{\nu} \cdot h_{+-}^{\eta})|^2 = a(1-a)^2 g_R^2 (1-b)^2, \\ |(l_{\bar{\nu}} \cdot h_{-+}^{\eta})|^2 &= a(g_R - g_L b [1 - (1-b)a])^2, \quad |(l_{\bar{\nu}} \cdot h_{+-}^{\eta})|^2 = a(1-a)^2 g_L^2 (1-b)^2. \end{aligned}$$

one can write these cross sections in a more “compact” form

$$\frac{d\tilde{\sigma}_{\text{inc}}^{v/\bar{v}}}{dT_A} = \sum_{f=p,n} \hat{F}_f \left[A_{\mp}^f g_{R/L}^2 (1-a)X + A_{\pm}^f Z(g_{L/R}, g_{R/L}) \right]. \quad (333)$$

Direct calculation⁴¹ for X and slightly longer calculation⁴² for $Z(P, Q)$ yield

$$\begin{aligned} X &= 1 - a + ab^2, \\ Z(P, Q) &= (P - Qab^2)^2 + Q^2 ab^2 (1-a). \end{aligned} \quad (334)$$

Using relations (334), incoherent $v/\bar{v}$ A scattering cross sections (333) take the form

$$\begin{aligned} \frac{d\tilde{\sigma}_{\text{inc}}^v}{dT_A} &= \sum_{f=p,n} F_f [A_{-}^f g_R^2 (1-a)(1-a+ab^2) + A_{+}^f \{(g_L - g_R ab^2)^2 + g_R^2 ab^2 (1-a)\}], \\ \frac{d\tilde{\sigma}_{\text{inc}}^{\bar{v}}}{dT_A} &= \sum_{f=p,n} F_f [A_{+}^f g_L^2 (1-a)(1-a+ab^2) + A_{-}^f \{(g_R - g_L ab^2)^2 + g_L^2 ab^2 (1-a)\}], \end{aligned}$$

which (considering that $g_{L,R} \equiv g_{L,R}^f$, $b \equiv b_f$) can be written in terms of the spin nuclear parameters $A_{\mp}^f$, form factor variable of nuclear structure (331), and kinematic relations (317) in the final form for this section,

$$\begin{aligned} \frac{d\tilde{\sigma}_{\text{inc}}^{v/\bar{v}}}{dT_A} &= \sum_{f=p,n} \hat{F}_f(q^2) \left[A_{\mp}^f (g_{R/L}^f)^2 (1-a)(1-a+ab_f^2) + A_{\pm}^f \left\{ (g_{L/R}^f - g_{R/L}^f ab_f^2)^2 + (g_{R/L}^f)^2 ab_f^2 (1-a) \right\} \right] \\ &= \sum_{f=p,n} \hat{F}_f(q^2) \left[A_{\mp}^f (g_{R/L}^f)^2 \frac{s(1-y)^2 - m^2(1-y)}{s-m^2} + A_{\pm}^f \left\{ \left(g_{L/R}^f - g_{R/L}^f \frac{ym^2}{s-m^2} \right)^2 + (g_{R/L}^f)^2 \frac{ym^2[s(1-y)-m^2]}{(s-m^2)^2} \right\} \right]. \end{aligned} \quad (335)$$

10.5.4. Cross section for incoherent (anti) neutrino scattering in terms of $\mathcal{A}$ and $\Delta\mathcal{A}$. In the expression for the incoherent scattering cross section (335) it is also worth going from the nuclear parameters $A_{\pm}$ to the nuclear parameters $\mathcal{A}^f$ and $\Delta\mathcal{A}^f$ defined in (321). Rearrangement (322) used above for the coherent case is now simpler,

$$\begin{aligned} \alpha_v &= (g_R^f)^2 (1-a)(1-a+ab_f^2), \\ \beta_v &= (g_L^f - g_R^f ab_f^2)^2 + (g_R^f)^2 ab_f^2 (1-a); \\ \alpha_{\bar{v}} &= (g_L^f)^2 (1-a)(1-a+ab_f^2), \\ \beta_{\bar{v}} &= (g_R^f - g_L^f ab_f^2)^2 + (g_L^f)^2 ab_f^2 (1-a). \end{aligned} \quad (337)$$

$$\begin{aligned} \frac{d\tilde{\sigma}_{\text{inc}}^v}{dT_A} &\propto A_{-}\alpha_v + A_{+}\beta_v = \frac{A-\Delta\mathcal{A}}{2}\alpha_v + \frac{A+\Delta\mathcal{A}}{2}\beta_v \\ &= \frac{A}{2}(\alpha_v + \beta_v) + \frac{\Delta\mathcal{A}}{2}(\beta_v - \alpha_v), \\ \frac{d\tilde{\sigma}_{\text{inc}}^{\bar{v}}}{dT_A} &\propto A_{+}\alpha_{\bar{v}} + A_{-}\beta_{\bar{v}} = \frac{A+\Delta\mathcal{A}}{2}\alpha_{\bar{v}} + \frac{A-\Delta\mathcal{A}}{2}\beta_{\bar{v}} \\ &= \frac{A}{2}(\alpha_{\bar{v}} + \beta_{\bar{v}}) + \frac{(-\Delta\mathcal{A})}{2}(\beta_{\bar{v}} - \alpha_{\bar{v}}), \end{aligned} \quad (336)$$

where other auxiliary variables are introduced,

The outlines of the general rule come out: to obtain the expression for the antineutrino $\frac{d\tilde{\sigma}_{\text{inc}}^{\bar{v}}}{dT_A}(\mathcal{A}^f, \Delta\mathcal{A}^f, \alpha_{\bar{v}}, \beta_{\bar{v}})$, make the replacements $\mathcal{A}^f \rightarrow \mathcal{A}^f$, $\Delta\mathcal{A}^f \rightarrow -\Delta\mathcal{A}^f$, $\alpha_v \rightarrow \alpha_{\bar{v}}$ and $\beta_v \rightarrow \beta_{\bar{v}}$ (or, which is equivalent, $g_L^f \leftrightarrow g_R^f$) in the formula for the neutrino $\frac{d\tilde{\sigma}_{\text{inc}}^v}{dT_A}(\mathcal{A}^f, \Delta\mathcal{A}^f, \alpha_v, \beta_v)$.

⁴¹ $X = 1 - 2(1-b)a + (1-b)^2 a^2 + (a-a^2)(1-b)^2 = 1 - 2a + 2ab + (a+2ab+ab^2) = 1 - a + ab^2$.

⁴² $Z(P, Q) = a(P - [Qb - Qb(1-b)a])^2 + (1-a)[P + Qb(1-b)a]^2 = (1-a)[P + Qab(1-b)]^2 + a[P + Qab(1-b) - Qb]^2$
 $= [P + Qab(1-b)]^2 - a[P + Qab(1-b)]^2 + a[P + Qab(1-b)]^2 + 2a[-P - Qab(1-b)]Qb + aQ^2b^2 = P^2 + 2PQab(1-b)$
 $+ Q^2a^2b^2(1-b)^2 - 2PQab - 2Q^2a^2b^2(1-b) + aQ^2b^2 = P^2 - 2PQab^2 + Q^2a^2b^2(1-2b+b^2) - 2Q^2a^2b^2(1-b)$
 $+ aQ^2b^2 + 2PQab - 2PQab = P^2 - 2PQab^2 + Q^2a^2b^2 + 2Q^2a^2b^3 + Q^2a^2b^2 + 2Q^2a^2b^3 + aQ^2b^2$
 $= \underbrace{P^2 - 2PQab^2 + Q^2a^2b^4 + Q^2a^2b^2 - 2Q^2a^2b^2}_{(P-Qab^2)^2} + aQ^2b^2 + \underbrace{2Q^2a^2b^3 - 2Q^2a^2b^3}_0 = (P - Qab^2)^2 + Q^2ab^2(1-a).$

Using kinematic relations from section 10.5 of the type $a = ys/(s - m)^2$, $ab^2 = ym^2/(s - m^2)$, and $1 - a + ab^2 = 1 - y$, for sums and differences of auxiliary variables (337) one obtains

$$\begin{aligned}\alpha_v + \beta_v &= g_L^2 - 2ab^2 g_L g_R + (1 - a + ab^2)^2 g_R^2 = g_L^2 - 2g_L g_R \frac{m^2 y}{s - m^2} + g_R^2 (1 - y)^2, \\ \beta_v - \alpha_v &= (g_L^f - g_R^f ab_f^2)^2 + (g_R^f)^2 ab_f^2 (1 - a) - (g_R^f)^2 (1 - a)^2 - (g_R^f)^2 (1 - a) ab_f^2 \\ &= (g_L^f - g_R^f ab_f^2 + g_R^f - ag_R^f)(g_L^f - g_R^f ab_f^2 - g_R^f + ag_R^f) = \left(g_L + g_R \left(1 - y \frac{s + m^2}{s - m^2} \right) \right) (g_L - g_R (1 - y)); \\ \alpha_{\bar{v}} + \beta_{\bar{v}} &= g_R^2 - 2g_R g_L \frac{m^2 y}{s - m^2} + g_L^2 (1 - y)^2, \quad \beta_{\bar{v}} - \alpha_{\bar{v}} = \left(g_R + g_L \left(1 - y \frac{s + m^2}{s - m^2} \right) \right) (g_R - g_L (1 - y)).\end{aligned}$$

Considering these expressions based on (335), the cross sections for incoherent (anti)neutrino–nucleus scattering in terms of weak chiral constants take the form

$$\begin{aligned}\frac{d\sigma_{\text{inc}}^v}{dT_A} &= \frac{2G_F^2 m_A}{\pi} \sum_{f=p,n} F_f(q^2) \left\{ A^f \left[g_L^2 - 2g_L^f g_R^f \frac{ym^2}{s - m^2} + g_R^2 (1 - y)^2 \right] \right. \\ &\quad \left. + \Delta A^f (g_L^f + g_R^f (y - 1)) \left[g_L^f + g_R^f \left(1 + y \frac{m^2 + s}{m^2 - s} \right) \right] \right\},\end{aligned}\tag{338}$$

$$\begin{aligned}\frac{d\sigma_{\text{inc}}^{\bar{v}}}{dT_A} &= \frac{2G_F^2 m_A}{\pi} \sum_{f=p,n} \hat{F}_f(q^2) \left\{ A^f \left[g_R^2 - 2g_L^f g_R^f \frac{ym^2}{s - m^2} + g_L^2 (1 - y)^2 \right] \right. \\ &\quad \left. - \Delta A^f (g_R^f + g_L^f (y - 1)) \left[g_R^f + g_L^f \left(1 + y \frac{m^2 + s}{m^2 - s} \right) \right] \right\}.\end{aligned}\tag{339}$$

Note that the first terms in square brackets proportional to A^f exactly correspond to the standard result for the scattering of neutrinos and antineutrinos by a point target (nucleon, electron) with the weak coupling constants g_L^f and g_R^f (see, for example, [42, 163]).

“Transition sums and differences” constructed from auxiliary variables (337) can be written in terms of the vector and axial-vector nucleon weak constants, considering that $g_L = \frac{1}{2}(g_V + g_A)$, $g_R = \frac{1}{2}(g_V - g_A)$,

$$\begin{aligned}\alpha_v + \beta_v &= \left[2 \left(1 - \frac{ys}{s - m^2} \right) + y^2 + y \frac{4m^2}{s - m^2} \right] \frac{g_A^2}{4} + 2y(2 - y) \frac{g_A^f g_V^f}{4} + \left[2 \left(1 - \frac{ys}{s - m^2} \right) + y^2 \right] \frac{g_V^2}{4}, \\ \beta_v - \alpha_v &= \frac{1}{4} \left[y \frac{s + m^2}{s - m^2} (g_A^f - g_V^f) + 2g_V^f \right] [(2 - y)g_A^f + yg_V^f], \\ \alpha_{\bar{v}} + \beta_{\bar{v}} &= \left[2 \left(1 - \frac{ys}{s - m^2} \right) + y^2 + y \frac{4m^2}{s - m^2} \right] \frac{g_A^2}{4} + 2y(2 - y) \frac{(-g_A^f)g_V^f}{4} + \left[2 \left(1 - \frac{ys}{s - m^2} \right) + y^2 \right] \frac{g_V^2}{4}, \\ \beta_{\bar{v}} - \alpha_{\bar{v}} &= \frac{1}{4} \left[y \frac{s + m^2}{s - m^2} (-g_A^f - g_V^f) + 2g_V^f \right] [-(2 - y)g_A^f + yg_V^f].\end{aligned}$$

Note that in this case, transition from “the neutrino” to “the antineutrino” is through the replacements $A^f \rightarrow A^f$, $\Delta A^f \rightarrow -\Delta A^f$, $g_V^f \leftrightarrow g_V^f$, and $g_A^f \leftrightarrow g_A^f$ (which is equivalent to $g_L^f \leftrightarrow g_L^f$).

Incoherent scattering cross sections (338) and (339) take the form

$$\begin{aligned}\frac{d\sigma_{\text{inc}}^v}{dT_A} &= \frac{G_F^2 m_A}{2\pi} \sum_{f=p,n} \hat{F}_f(q^2) \left\{ A^f \left[\left(2 \left(1 - \frac{ys}{s - m^2} \right) + y^2 + y \frac{4m^2}{s - m^2} \right) (g_A^f)^2 + g_A^f g_V^f 2y(2 - y) \right. \right. \\ &\quad \left. \left. + \left(2 \left(1 - \frac{ys}{s - m^2} \right) + y^2 \right) (g_V^f)^2 \right] + \Delta A^f \left[2g_V^f - y \frac{s + m^2}{s - m^2} (g_V^f - g_A^f) \right] [yg_V^f + g_A^f (2 - y)] \right\}, \\ \frac{d\sigma_{\text{inc}}^{\bar{v}}}{dT_A} &= \frac{G_F^2 m_A}{2\pi} \sum_{f=p,n} \hat{F}_f(q^2) \left\{ A^f \left[\left(2 \left(1 - \frac{ys}{s - m^2} \right) + y^2 + y \frac{4m^2}{s - m^2} \right) (g_A^f)^2 - g_A^f g_V^f 2y(2 - y) \right. \right. \\ &\quad \left. \left. + \left(2 \left(1 - \frac{ys}{s - m^2} \right) + y^2 \right) (g_V^f)^2 \right] - \Delta A^f \left[2g_V^f - y \frac{s + m^2}{s - m^2} (g_V^f + g_A^f) \right] [yg_V^f - g_A^f (2 - y)] \right\}.\end{aligned}$$

These formulas for the incoherent (anti)neutrino–nucleus scattering cross section (in terms of g_V^f and g_A^f) do not seem preferable to formulas (338) and (339) above.

Expression (338) for the incoherent neutrino–nucleus scattering cross section corresponds to the formula for incoherent νA scattering (46) from [19] up to redefinition of the correction factor g_{inc}^f entering into the form factor $\hat{F}_f(q^2) = g_{inc}^f(1 - |F_f(q^2)|^2)$ [19].

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REFERENCES

1. D. Z. Freedman, “Coherent effects of a weak neutral current,” *Phys. Rev. D* **9**, 1389–1392 (1974).
2. V. B. Kopeliovich and L. L. Frankfurt, “Isotopic and chiral structure of neutral current,” *JETP Lett.* **19**, 145–147 (1974).
3. D. Z. Freedman, D. N. Schramm, and D. L. Tubbs, “The weak neutral current and its effects in stellar collapse,” *Ann. Rev. Nucl. Part. Sci.* **27**, 167–207 (1977).
4. Yu. V. Gaponov and V. N. Tikhonov, “Elastic scattering of low-energy neutrinos by atomic systems,” *Yad. Fiz.* **26**, 594–600 (1977).
5. L. M. Sehgal and M. Wanninger, “Atomic effects in coherent neutrino scattering,” *Phys. Lett. B* **171**, 107–112 (1986).
6. M. Cadeddu, F. Dordei, C. Giunti, K. A. Kouzakov, E. Picciau, and A. I. Studenikin, “Potentialities of a low-energy detector based on ^4He evaporation to observe atomic effects in coherent neutrino scattering and physics perspectives,” [arXiv]1907.03302.
7. J. Barranco, O. G. Miranda, and T. I. Rashba, “Probing new physics with coherent neutrino scattering off nuclei,” *J. High Energy Phys.*, No. 12, 21 (2005); [arXiv] hep-ph/0508299.
8. D. Akimov et al. (COHERENT Collab.), “Observation of coherent elastic neutrino-nucleus scattering,” *Science* **357**, 1123–1126 (2017); [arXiv]1708.01294.
9. D. Akimov et al. (COHERENT Collab.), “The COHERENT experiment at the spallation neutron source,” [arXiv]1509.08702.
10. D. Akimov et al. (COHERENT Collab.), “COHERENT 2018 at the spallation neutron source,” [arXiv]1803.09183.
11. D. Akimov et al. (COHERENT Collab.), “COHERENT Collaboration data release from the first observation of coherent elastic neutrino-nucleus scattering,” [arXiv]1804.09459.
12. Scholberg K. (COHERENT Collab.), “Observation of coherent elastic neutrino-nucleus scattering by COHERENT,” *PoS NuFact2017*, 020 (2018); [arXiv]1801.05546.
13. D. Akimov et al. (COHERENT Collab.), “Observation of coherent elastic neutrino-nucleus scattering,” *Science* **357**, 1123–1126 (2017); [arXiv]1708.01294.
14. J. I. Collar, N. E. Fields, M. Hai, T. W. Hossbach, J. L. Orrell, C. T. Overman, G. Perumpilly, and B. Scholz, “Coherent neutrino-nucleus scattering detection with a CsI[Na] scintillator at the SNS spallation source,” *Nucl. Instrum. Methods Phys. Res., Sect. A* **773**, 56–65 (2015); [arXiv]1407.7524.
15. D. Akimov et al. (CSI Collab.), “Coherent scattering investigations at the spallation neutron source: A Snowmass white paper,” [arXiv]1310.0125.
16. A. Bolozdynya et al., “Opportunities for neutrino physics at the spallation neutron source: A white paper,” [arXiv]1211.5199.
17. K. Scholberg, “Prospects for measuring coherent neutrino-nucleus elastic scattering at a stopped-pion neutrino source,” *Phys. Rev. D* **73**, 033005 (2006); [arXiv] hep-ex/0511042.
18. D. Akimov et al. (COHERENT Collab.), “First detection of coherent elastic neutrino-nucleus scattering on argon,” [arXiv]2003.10630.
19. V. A. Bednyakov and D. V. Naumov, “Coherency and incoherency in neutrino-nucleus elastic and inelastic scattering,” *Phys. Rev. D* **98**, 053004 (2018); [arXiv]1806.08768.
20. V. A. Bednyakov and D. V. Naumov, “On coherent neutrino and antineutrino scattering off nuclei,” *Phys. Part. Nucl. Lett.* **16**, 638–646 (2019); [arXiv]1904.03119.
21. T. W. Donnelly and J. D. Walecka, “Electron scattering and nuclear structure,” *Annu. Rev. Nucl. Part. Sci.* **25**, 329–405 (1975).
22. J. Bernabeu, “Low-energy elastic neutrino-nucleon and nuclear scattering and its relevance for supernovae,” Preprint CERN-TH-2073 (CERN, Geneva, 1975).
23. P. S. Amanik and G. C. McLaughlin, “Neutron form factor from neutrino-nucleus coherent elastic scattering,” [arXiv]0707.4191.
24. C. J. Horowitz, K. J. Coakley, and D. N. McKinsey, “Supernova observation via neutrino-nucleus elastic scattering in the CLEAN detector,” *Phys. Rev. D* **68**, 023005 (2003); [arXiv]astro-ph/0302071.
25. J. Engel, “Nuclear form factors for the scattering of weakly interacting massive particles,” *Phys. Lett. B* **264**, 114–119 (1991).
26. P. S. Amanik and G. C. McLaughlin, “Nuclear neutron form factor from neutrino nucleus coherent elastic scattering,” *J. Phys. G* **36**, 015105 (2009).
27. E. Akhmedov, G. Arcadi, M. Lindner, and S. Vogl, “Coherent scattering and macroscopic coherence: Implications for neutrino, dark matter and axion detection,” *J. High Energy Phys.*, No. 10, 45 (2018); [arXiv]1806.10962.
28. Yu. V. Gaponov, Yu. L. Dobrynin, and V. N. Tikhonov, “Neutrino and anti-neutrino scattering on hydrogen-like atom,” *Yad. Fiz.* **22**, 328–334 (1975).

29. J. Weber, “Gravitons, neutrinos, and anti-neutrinos,” *Found. Phys.* **14**, 1185–1209 (1984).
30. J. Weber, “Method for observation of neutrinos and antineutrinos,” *Phys. Rev. C* **31**, 1468–1475 (1985).
31. J. Weber, “Apparent observation of abnormally large coherent scattering cross-sections using keV and MeV range anti-neutrinos, and solar neutrinos,” *Phys. Rev. D* **38**, 32–39 (1988).
32. L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media*, 2nd ed. (Pergamon Press, Oxford, 1984) Vol. 8, Secs. 124 and 125.
33. P. F. Smith, “Coherent neutrino scattering – relativistic and nonrelativistic,” *Nuovo Cimento A* **83**, 263–274 (1984).
34. S. Kerman, V. Sharma, M. Deniz, H. T. Wong, J. W. Chen, H. B. Li, S. T. Lin, C.-P. Liu, and Q. Yue (TEXONO Collab.), “Coherency in neutrino-nucleus elastic scattering,” *Phys. Rev. D* **93**, 113006 (2016); [arXiv]1603.08786.
35. D. K. Papoulias and T. S. Kosmas, “Standard and nonstandard neutrino-nucleus reactions cross sections and event rates to neutrino detection experiments,” *Adv. High Energy Phys.* **2015**, 763648 (2015); [arXiv]1502.02928.
36. D. K. Papoulias and T. S. Kosmas, “Nuclear aspects of neutral current non-standard ν -nucleus reactions and the role of the exotic $\mu^- \rightarrow e^-$ transitions experimental limits,” *Phys. Lett. B* **728**, 482–488 (2014); [arXiv]1312.2460.
37. P. Vogel and J. Engel, “Neutrino electromagnetic form factors,” *Phys. Rev. D* **39**, 3378 (1989).
38. P. C. Divari, “Coherent and incoherent neutral current scattering for supernova detection,” *Adv. High Energy Phys.* **2012**, 379460 (2012).
39. P. Pirinen, J. Suhonen, and E. Ydrefors, “Neutral-current neutrino-nucleus scattering off Xe isotopes,” [arXiv]1804.08995.
40. V. Tsakstara, T. Kosmas, and J. Vergados, “Weak responses of neutral current neutrino-nucleus reactions,” *Rom. J. Phys.* **58**, 1258–1269 (2013).
41. M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory* (Addison-Wesley, Reading, 1995).
42. S. M. Bilenky, *Introduction to Feynman Diagrams and Electroweak Interactions Physics* (Ed. Frontieres, Gif-sur-Yvette, 1995).
43. A. Drukier and L. Stodolsky, “Principles and applications of a neutral current detector for neutrino physics and astronomy,” *Phys. Rev. D* **30**, 2295 (1984).
44. K. Patton, J. Engel, G. C. McLaughlin, and N. Schunck, “Neutrino-nucleus coherent scattering as a probe of neutron density distributions,” *Phys. Rev. C* **86**, 024612 (2012); [arXiv]1207.0693.
45. N. Jachowicz, K. Heyde, and S. Rombouts, “Many body description of neutrino nucleus interactions,” *Nucl. Phys. A* **688**, 593–595 (2001).
46. P. C. Divari, V. C. Chasioti, and T. S. Kosmas, “Neutral current neutrino-Mo-98 reaction cross sections at low and intermediate energies,” *Phys. Scripta* **82**, 065201 (2010).
47. G. McLaughlin, “Theory and phenomenology of coherent neutrino-nucleus scattering,” *AIP Conf. Proc.* **1666**, 160001 (2015).
48. J. D. Vergados, F. T. Avignone, III, and I. Giomataris, “Coherent neutral current neutrino-nucleus scattering at a spallation source: A valuable experimental probe,” *Phys. Rev. D* **79**, 113001 (2009); [arXiv]0902.1055.
49. J. Papavassiliou, J. Bernabeu, and M. Passera, “Neutrino-nuclear coherent scattering and the effective neutrino charge radius,” *PoS HEP2005*, 192 (2006); [arXiv]hep-ph/0512029.
50. V. V. Belov et al., “Investigation of neutrino properties with the low-background germanium spectrometer gemma-iii,” in *JINR Program Advisory Committee*, 2018.
51. D. Barker and D. M. Mei, “Germanium detector response to nuclear recoils in searching for dark matter,” *Astropart. Phys.* **38**, 1–6 (2012); [arXiv]1203.4620.
52. V. Belov et al., “The NuGeN experiment at the Kalinin Nuclear Power Plant,” *J. Instrum.* **10**, P12011 (2015).
53. P. Agnes et al. (DarkSide Collab.), “Low-mass dark matter search with the DarkSide-50 experiment,” [arXiv]1802.06994.
54. J. Piekarewicz, A. R. Linero, P. Giuliani, and E. Chicken, “Power of two: Assessing the impact of a second measurement of the weak-charge form factor of ^{208}Pb ,” *Phys. Rev. C* **94**, 034316 (2016); [arXiv]1604.07799.
55. R. H. Helm, “Inelastic and elastic scattering of 187-MeV electrons from selected even-even nuclei,” *Phys. Rev. D* **104**, 1466–1475 (1956).
56. M. Cadeddu and F. Dordei, “Reinterpreting the weak mixing angle from atomic parity violation in view of the Cs neutron rms radius measurement from COHERENT,” *Phys. Rev. D* **99**, 033010 (2019); [arXiv]1808.10202.
57. C. Boehm, D. G. Cerdeño, P. A. N. Machado, A. O.-D. Campo, and E. Reid, “How high is the neutrino floor?” *J. Cosmol. Astropart. Phys.* **1901**, 043 (2019); [arXiv]1809.06385.
58. V. Brdar, W. Rodejohann, and X.-J. Xu, “Producing a new fermion in coherent elastic neutrino-nucleus scattering: From neutrino mass to dark matter,” *J. High Energy Phys.*, No. **12**, 24 (2018); [arXiv]1810.03626.
59. M. Cadeddu, C. Giunti, K. A. Kouzakov, Y. F. Li, A. I. Studenikin, and Y. Y. Zhang, “Neutrino charge radii from COHERENT elastic neutrino-nucleus scattering,” *Phys. Rev. D* **98**, 113010 (2018); [arXiv]1810.05606.
60. A. Millar, G. Raffelt, L. Stodolsky, and E. Vitagliano, “Neutrino mass from bremsstrahlung endpoint in coherent scattering on nuclei,” *Phys. Rev. D* **98**, 123006 (2018); [arXiv]1810.06584.
61. W. Altmannshofer, M. Tammaro, and J. Zupan, “Non-standard neutrino interactions and low energy experiments,” [arXiv]1812.02778.
62. C. Blanco, D. Hooper, and P. Machado, “Constraining sterile neutrino interpretations of the LSND and MiniBooNE anomalies with coherent neutrino scattering experiments,” [arXiv]1901.08094.
63. D. Aristizabal Sierra, J. Liao, and D. Marfatia, “Impact of form factor uncertainties on interpretations of

- coherent elastic neutrino-nucleus scattering data,” [arXiv]1902.07398.
64. X.-R. Huang and L.-W. Chen, “Neutron skin in CsI and low-energy effective weak mixing angle from COHERENT data,” [arXiv]1902.07625.
 65. O. G. Miranda, G. Sanchez Garcia, and O. Sanders, “Testing new physics with future COHERENT experiments,” [arXiv]1902.09036.
 66. D. K. Papoulias, T. S. Kosmas, R. Sahu, V. K. B. Kota, and M. Hota, “Constraining nuclear physics parameters with current and future COHERENT data,” [arXiv]1903.03722.
 67. J. I. Collar, A. R. L. Kavner, and C. M. Lewis, “Response of CsI[Na] to nuclear recoils: Impact on coherent elastic neutrino-nucleus scattering (CEvNS),” [arXiv]1907.04828.
 68. D. K. Papoulias, “COHERENT constraints after the Chicago-3 quenching factor measurement,” [arXiv]1907.11644.
 69. A. N. Khan and W. Rodejohann, “New physics from COHERENT data with improved quenching factors,” [arXiv]1907.12444.
 70. D. V. Naumov and V. A. Naumov, “A diagrammatic treatment of neutrino oscillations,” *J. Phys. G* **37**, 105014 (2010); [arXiv]1008.0306.
 71. D. V. Naumov, “On the theory of wave packets,” *Phys. Part. Nucl. Lett.* **10**, 642–650 (2013); [arXiv]1309.1717.
 72. D. Karlovets, “Scattering of wave packets with phases,” *J. High Energy Phys.*, No. **03**, 49 (2017); [arXiv]1611.08302.
 73. J. R. Wilson, “Coherent neutrino scattering and stellar collapse,” *Phys. Rev. Lett.* **32**, 849–852 (1974).
 74. S. Rombouts and K. Heyde, “Calculation of neutrino nucleus scattering cross-sections relevant for supernova processes,” *Nucl. Phys. A* **621**, 371–374 (1997).
 75. E. P. O’Connor and S. M. Couch, “Two dimensional core-collapse supernova explosions aided by general relativity with multidimensional neutrino transport,” *Astrophys. J.* **854**, 63 (2018); [arXiv]1511.07443.
 76. Y. Giomataris and J. D. Vergados, “A network of neutral current spherical TPC’s for dedicated supernova detection,” *Phys. Lett. B* **634**, 23–29 (2006); [arXiv]hep-ex/0503029.
 77. J. D. Vergados and Y. Giomataris, “Neutral current coherent cross sections. Implications on detecting SN and earth neutrinos with gaseous spherical TPC,” *Int. J. Mod. Phys.* 2017. V. E26, no. 01n02, P. 1740030, [arXiv]1603.03966.
 78. P. C. Divari, S. Galanopoulos, and G. A. Souliotis, “Coherent scattering of neutral-current neutrinos as a probe for supernova detection,” *J. Phys. G* **39**, 095204 (2012).
 79. G. I. Lykasov and V. A. Bednyakov, “Neutrino-nucleus interactions at low energies within Fermi-liquid theory,” *Phys. Rev.* **76**, 014622 (2007); [arXiv]nucl-th/0703036.
 80. L. Coraggio, L. De Angelis, T. Fukui, A. Gargano, and N. Itaco, “Calculation of Gamow-Teller and two-neutrino double- β decay properties for ^{130}Te and ^{136}Xe with a realistic nucleon-nucleon potential,” *Phys. Rev. C* **95**, 064324 (2017); [arXiv]1703.05087.
 81. R. H. Helm, “Inelastic and elastic scattering of 187-MeV electrons from selected even-even nuclei,” *Phys. Rev.* **104**, 1466–1475 (1956).
 82. D. W. L. Sprung and J. Martorell, “The symmetrized Fermi function and its transforms,” *J. Phys. A: Math. Gen.* **30**, 6525–6534 (1997).
 83. S. Klein and J. Nystrand, “Exclusive vector meson production in relativistic heavy ion collisions,” *Phys. Rev. C* **60**, 014903 (1999); [arXiv]hep-ph/9902259.
 84. M. Cadeddu, C. Giunti, Y. F. Li, and Y. Zhang, “Average CsI neutron density distribution from COHERENT data,” [arXiv]1710.02730.
 85. E. Ciuffoli, “Measuring the neutron distribution from coherent elastic neutrino nucleus scattering,” [arXiv]1903.12120.
 86. I. Sick, “Proton charge radius from electron scattering,” *Atoms* **6**, 2 (2018); [arXiv]1801.01746.
 87. M. Baldo and G. F. Burgio, “The nuclear symmetry energy,” *Prog. Part. Nucl. Phys.* **91**, 203–258 (2016); [arXiv]1606.08838.
 88. M. Cadeddu, F. Dordei, C. Giunti, Y. F. Li, and Y. Y. Zhang, “Neutrino, electroweak and nuclear physics from COHERENT elastic neutrino-nucleus scattering with refined quenching factor,” [arXiv]1908.06045.
 89. C. Giusti and M. V. Ivanov, “Neutral current neutrino-nucleus scattering theory,” [arXiv]1908.08603.
 90. C. G. Payne, S. Bacca, G. Hagen, W. Jiang, and T. Papenbrock, “Coherent elastic neutrino-nucleus scattering on ^{40}Ar from first principles,” [arXiv]1908.09739.
 91. R. Bernabei et al., “First model independent results from DAMA/LIBRA-Phase2,” *Universe* **4**, 116 (2018); [arXiv]1805.10486; *Nucl. Phys. At. Energy* **19**, 307 (2018).
 92. R. Bernabei et al., “Improved model-dependent corollary analyses after the first six annual cycles of DAMA/LIBRA-phase2,” [arXiv]1907.06405.
 93. J. Billard, L. Strigari, and E. Figueroa-Feliciano, “Implication of neutrino backgrounds on the reach of next generation dark matter direct detection experiments,” *Phys. Rev. D* **89**, 023524 (2014); [arXiv]1307.5458.
 94. V. A. Bednyakov, “Is it possible to discover a dark matter particle with an accelerator?” *Phys. Part. Nucl.* **47**, 711–774 (2016); [arXiv]1505.04380.
 95. A. Gutlein et al., “Impact of coherent neutrino nucleus scattering on direct dark matter searches based on CaWO_4 crystals,” *Astropart. Phys.* **69**, 44–49 (2015); [arXiv]1408.2357.
 96. A. J. Anderson, J. M. Conrad, E. Figueroa-Feliciano, K. Scholberg, and J. Spitz, “Coherent neutrino scattering in dark matter detectors,” *Phys. Rev. D* **84**, 013008 (2011); [arXiv]1103.4894.
 97. P. deNiverville, M. Pospelov, and A. Ritz, “Light new physics in coherent neutrino-nucleus scattering experiments,” *Phys. Rev. D* **92**, 095005 (2015); [arXiv]1505.07805.
 98. S.-F. Ge and I. M. Shoemaker, “Constraining photon portal dark matter with Texono and Coherent data,” [arXiv]1710.10889.

99. C. Giunti and A. Studenikin, “Neutrino electromagnetic interactions: A window to new physics,” *Rev. Mod. Phys.* **87**, 531 (2015); [arXiv]1403.6344.
100. K. A. Kouzakov and A. I. Studenikin, “Electromagnetic properties of massive neutrinos in low-energy elastic neutrino-electron scattering,” *Phys. Rev. D* **95**, 055013 (2017); [arXiv]1703.00401; Erratum: *Phys. Rev. D* **96**, 099904 (2017).
101. D. K. Papoulias and T. S. Kosmas, “COHERENT constraints to conventional and exotic neutrino physics,” *Phys. Rev. D* **97**, 033003 (2018); [arXiv]1711.09773.
102. H. Novales-Sanchez, A. Rosado, V. Santiago-Olan, and J. J. Toscano, “Effects of physics beyond the standard model on the neutrino charge radius: An effective Lagrangian approach,” *Phys. Rev. D* **78**, 073014 (2008); [arXiv]0805.4177.
103. A. J. Anderson, J. M. Conrad, E. Figueroa-Feliciano, C. Ignarra, G. Karagiorgi, K. Scholberg, M. H. Shaevitz, and J. Spitz, “Measuring active-to-sterile neutrino oscillations with neutral current coherent neutrino-nucleus scattering,” *Phys. Rev. D* **86**, 013004 (2012); [arXiv]1201.3805.
104. O. G. Miranda, D. K. Papoulias, M. Tortola, and J. W. F. Valle, “Probing neutrino transition magnetic moments with coherent elastic neutrino-nucleus scattering,” [arXiv]1905.03750.
105. R. E. Shrock, “Electromagnetic Properties and decays of Dirac and Majorana neutrinos in a general class of gauge theories,” *Nucl. Phys. B* **206**, 359–379 (1982).
106. W. Grimus, M. Maltoni, T. Schwetz, M. A. Tortola, and J. W. F. Valle, “Constraining Majorana neutrino electromagnetic properties from the LMA-MSW solution of the solar neutrino problem,” *Nucl. Phys. B* **648**, 376–396 (2003); [arXiv]hep-ph/0208132.
107. B. Dutta, S. Liao, S. Sinha, and L. E. Strigari, “Searching for beyond the standard model physics with COHERENT energy and timing data,” [arXiv]1903.10666.
108. T. S. Kosmas, O. G. Miranda, D. K. Papoulias, M. Tortola, and J. W. F. Valle, “Probing neutrino magnetic moments at the Spallation Neutron Source facility,” *Phys. Rev. D* **92**, 013011 (2015); [arXiv]1505.03202.
109. T. S. Kosmas, O. G. Miranda, D. K. Papoulias, M. Tortola, Valle J. W. F. and, “Sensitivities to neutrino electromagnetic properties at the TEXONO experiment,” *Phys. Lett. B* **750**, 459–465 (2015); [arXiv]1506.08377.
110. M. Tanabashi et al. (Particle Data Group Collab.), *Review of Particle Physics*, *Phys. Rev. D* **98**, 030001 (2018).
111. J. Billard, J. Johnston, and B. J. Kavanagh, “Prospects for exploring new physics in coherent elastic neutrino-nucleus scattering,” [arXiv]1805.01798.
112. B. C. Canas, E. A. Garces, O. G. Miranda, M. Tortola, and J. W. F. Valle, “The weak mixing angle from low energy neutrino measurements: A global update,” *Phys. Lett. B* **761**, 450–455 (2016); [arXiv]1608.02671.
113. B. C. Canas, E. A. Garces, O. G. Miranda, and A. Parada, “Future perspectives for a weak mixing angle measurement in coherent elastic neutrino nucleus scattering experiments,” [arXiv]1806.01310.
114. G. Arcadi, M. Lindner, J. Martins, and F. S. Queiroz, “New physics probes: Atomic parity violation, polarized electron scattering and neutrino-nucleus coherent scattering,” [arXiv]1906.04755.
115. I. Esteban, M. C. Gonzalez-Garcia, M. Maltoni, I. Martinez-Soler, and J. Salvado, “Updated constraints on non-standard interactions from global analysis of oscillation data,” [arXiv]1805.04530.
116. J. B. Dent, B. Dutta, S. Liao, J. L. Newstead, L. E. Strigari, and J. W. Walker, “Accelerator and reactor complementarity in coherent neutrino scattering,” [arXiv]1711.03521.
117. M. Abdullah, J. B. Dent, B. Dutta, G. L. Kane, S. Liao, and L. E. Strigari, “Coherent elastic neutrino nucleus scattering (CEvNS) as a probe of Z' through kinetic and mass mixing effects,” [arXiv]1803.01224.
118. J. Liao and D. Marfatia, “COHERENT constraints on nonstandard neutrino interactions,” [arXiv]1708.04255.
119. D. Aristizabal Sierra, V. De Romeri, and N. Rojas, “COHERENT analysis of neutrino generalized interactions,” [arXiv]1806.07424.
120. Y. Farzan and M. Tortola, “Neutrino oscillations and non-standard interactions,” *Front. Phys.* **6**, 10 (2018); [arXiv]1710.09360.
121. M. Lindner, W. Rodejohann, and X.-J. Xu, “Coherent neutrino-nucleus scattering and new neutrino interactions,” *J. High Energy Phys.*, No. 03, 97 (2017); [arXiv]1612.04150.
122. Y. Farzan, M. Lindner, W. Rodejohann, and X.-J. Xu, “Probing neutrino coupling to a light scalar with coherent neutrino scattering,” *J. High Energy Phys.*, No. 05, 66 (2018); [arXiv]1802.05171.
123. J. Hakenmuller et al., “Neutron-induced background in the CONUS experiment,” [arXiv]1903.09269.
124. P. B. Denton, Y. Farzan, and I. M. Shoemaker, “A plan to rule out large non-standard neutrino interactions after COHERENT data” [arXiv]1804.03660.
125. D. Aristizabal Sierra, V. De Romeri, and N. Rojas, “CP violating effects in coherent elastic neutrino-nucleus scattering processes,” [arXiv]1906.01156.
126. C. Giunti, “General COHERENT constraints on neutrino nonstandard interactions,” *Phys. Rev. D* **101**, 035039 (2020); [arXiv]1909.00466.
127. A. Parada, “Constraints on neutrino electric milli-charge from experiments of elastic neutrino-electron interaction and future experimental proposals involving coherent elastic neutrino-nucleus scattering,” [arXiv]1907.04942.
128. B. Canas, E. Garces, O. Miranda, A. Parada, and G. Sanchez Garcia, “Interplay between nonstandard and nuclear constraints in coherent elastic neutrino-nucleus scattering experiments,” *Phys. Rev. D* **101**, 035012 (2020); [arXiv]1911.09831.
129. B. Dutta, R. F. Lang, S. Liao, S. Sinha, L. Strigari, and A. Thompson, “A global analysis strategy to resolve neutrino NSI degeneracies with scattering and oscillation data,” [arXiv]2002.03066.
130. W.-F. Chang, and J. Liao, “Constraints on light singlet fermion interactions from coherent elastic neutrino-nucleus scattering,” [arXiv]2002.10275.
131. O. Miranda, D. Papoulias, M. Tórtola, and J. Valle, “Probing new neutral gauge bosons with CEvNS and neutrino-electron scattering,” *Phys. Rev. D* **101**, 073005 (2020); [arXiv]2002.01482.

132. L. Flores, N. Nath, and E. Peinado, “Non-standard neutrino interactions in U(1)’ model after COHERENT data,” [arXiv]2002.12342.
133. M. Abdullah, D. Aristizabal Sierra, B. Dutta, and L. E. Strigari, “Coherent elastic neutrino-nucleus scattering with directional detectors,” [arXiv] 2003.11510.
134. O. Miranda, D. Papoulias, G. S. Garcia, O. Sanders, M. Tórtola, and J. Valle, “Implications of the first detection of coherent elastic neutrino-nucleus scattering (CEvNS) with liquid argon,” [arXiv]2003.12050.
135. M. Cadeddu, F. Dordei, C. Giunti, Y. Li, E. Picciau, and Y. Zhang, “Physics results from the first COHERENT observation of CEvNS in argon and their combination with cesium-iodide data,” [arXiv]2005.01645.
136. T. Li, X.-D. Ma, and M. A. Schmidt, “Generic neutrino interactions with sterile neutrinos in light of neutrino-nucleus coherent scattering and meson invisible decays,” [arXiv]2005.01543.
137. X. Qian and J.-C. Peng, “Physics with reactor neutrinos,” *Rep. Prog. Phys.* **82**, 036201 (2019); [arXiv] 1801.05386.
138. J. Sinatkas, V. Tsaktsara, and O. Kosmas, “Simulated neutrino signals of low and intermediate energy neutrinos on Cd detectors,” [arXiv]1904.01056.
139. M. F. Carneiro, “Neutrino cross sections: Status and prospects,” in *Proceedings of the Prospects in Neutrino Physics, NuPhys2017* (London, UK, 2017), pp. 13–22; [arXiv]1804.03238.
140. K. Mahn, C. Marshall, and C. Wilkinson, “Progress in measurements of 0.1–10 GeV neutrino–nucleus scattering and anticipated results from future experiments,” *Ann. Rev. Nucl. Part. Sci.* **68**, 105–129 (2018); [arXiv]1803.08848.
141. D. Baxter et al., “Coherent elastic neutrino-nucleus scattering at the European Spallation Source,” *J. High Energy Phys.*, No. **02**, 123 (2020); [arXiv]1911.00762.
142. Wong H. T., Li H.-B., Li J., Yue Q., and Zhou Z.-Y., “Research program towards observation of neutrino-nucleus coherent scattering,” *J. Phys. Conf. Ser.* **39**, 266–268 (2006); [arXiv]hep-ex/0511001.
143. H. T. Wong, “Neutrino-nucleus coherent scattering and dark matter searches with sub-keV germanium detector,” *Nucl. Phys. A* **844**, 229C–233C (2010).
144. D. Yu. Akimov et al. (RED Collab.), “Prospects for observation of neutrino-nuclear neutral current coherent scattering with two-phase Xenon emission detector,” *J. Instrum.* **8**, P10023 (2013); [arXiv]1212.1938.
145. Akimov D. Yu. et al., “The RED-100 two-phase emission detector,” *Instrum. Exp. Tech.* **60**, 175–181 (2017).
146. A. V. Kopylov, I. V. Orekhov, V. V. Petukhov, and A. E. Solomatin, “Gaseous detector of ionizing radiation for registration of coherent neutrino scattering on nuclei,” *Tech. Phys. Lett.* **40**, 185–187 (2014).
147. J. A. Formaggio, E. Figueroa-Feliciano, and A. J. Anderson, “Sterile neutrinos, coherent scattering and oscillometry measurements with low-temperature bolometers,” *Phys. Rev. D* **85**, 013009 (2012); [arXiv] 1107.3512.
148. S. J. Brice et al., “A method for measuring coherent elastic neutrino-nucleus scattering at a far off-axis high-energy neutrino beam target,” *Phys. Rev. D* **89**, 072004 (2014); [arXiv]1311.5958.
149. G. Fernandez Moroni, J. Estrada, E. E. Paolini, G. Cencelo, J. Tiffenberg, and J. Molina, “Charge coupled devices for detection of coherent neutrino-nucleus scattering,” *Phys. Rev. D* **91**, 072001 (2015); [arXiv]1405.5761.
150. A. Aguilar-Arevalo et al. (CONNIE Collab.), “The CONNIE experiment,” *J. Phys. Conf. Ser.* **761**, 012057 (2016); [arXiv]1608.01565.
151. A. Aguilar-Arevalo et al. (CONNIE Collab.), “Exploring low-energy neutrino physics with the Coherent Neutrino Nucleus Interaction Experiment (CONNIE),” [arXiv]1906.02200.
152. J. Billard et al., “Coherent neutrino scattering with low temperature bolometers at Chooz reactor complex,” *J. Phys. G* **44**, 105101 (2017); [arXiv]1612.09035.
153. R. Strauss et al., “The v-cleus experiment: A gram-scale fiducial-volume cryogenic detector for the first detection of coherent neutrino-nucleus scattering,” *Eur. Phys. J. C* **77**, 506 (2017); [arXiv]1704.04320.
154. L. Oberauer, “Coherent neutrino nucleus scattering,” *Prog. Part. Nucl. Phys.* **48**, 301–304 (2002).
155. S. Fallows, T. Kozynets, and C. B. Krauss, “Sensitivity of the PICO-500 bubble chamber to supernova neutrinos through coherent nuclear elastic scattering,” [arXiv] 1806.01417.
156. Agnolet G. et al. (MINER Collab.), “Background studies for the MINER coherent neutrino scattering reactor experiment,” *Nucl. Instrum. Methods Phys. Res., Sect. A* **853**, 53–60 (2017); [arXiv]1609.02066.
157. S. Sangiorgio, A. Bernstein, J. Coleman, M. Foxe, C. Hagmann, T. H. Joshi, I. Jovanovic, K. Kazkaz, K. Movrokoridis, and S. Pereverzev, “R&D for the observation of coherent neutrino-nucleus scatter at a nuclear reactor with a dual-phase argon ionization detector,” *Phys. Procedia* **37**, 1266–1272 (2012).
158. M. Lindner, “The CONUS coherent neutrino scattering experiment,” Presented at the Conference on Neutrino and Nuclear Physics, 15–20 October 2017, Catania, Italy.
159. C. Bellenghi, D. Chiesa, L. Di Noto, M. Pallavicini, E. Previtali, and M. Vignati, “Coherent elastic nuclear scattering of ^{51}Cr neutrinos,” [arXiv]1905.10611.
160. Q. Arnaud et al. (EDELWEISS Collab.), “Optimizing EDELWEISS detectors for low-mass WIMP searches,” *Phys. Rev. D* **97**, 022003 (2018); [arXiv] 1707.04308.
161. Armengaud E. et al. (EDELWEISS Collab.), “Searching for low-mass dark matter particles with a massive Ge bolometer operated above-ground,” *Phys. Rev. D* **99**, 082003 (2019); [arXiv]1901.03588.
162. R. R. Lewis, “Coherent detector for low-energy neutrinos,” *Phys. Rev. D* **21**, 663 (1980).
163. S. Bilenky, “Introduction to the physics of massive and mixed neutrinos,” *Lect. Notes Phys.* **947**, 1–273 (2018).

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